
11th Russbach School on Nuclear Astrophysics
9-15 March 2014 Russbach (Austria)

Dense (hadronic and quark) Matter

J. R. Stone,

University of Oxford/Tennessee/Maryland

Outline:

1. Introduction to nuclear matter

2. The Equation of State

2. Microscopic vs empirical approach

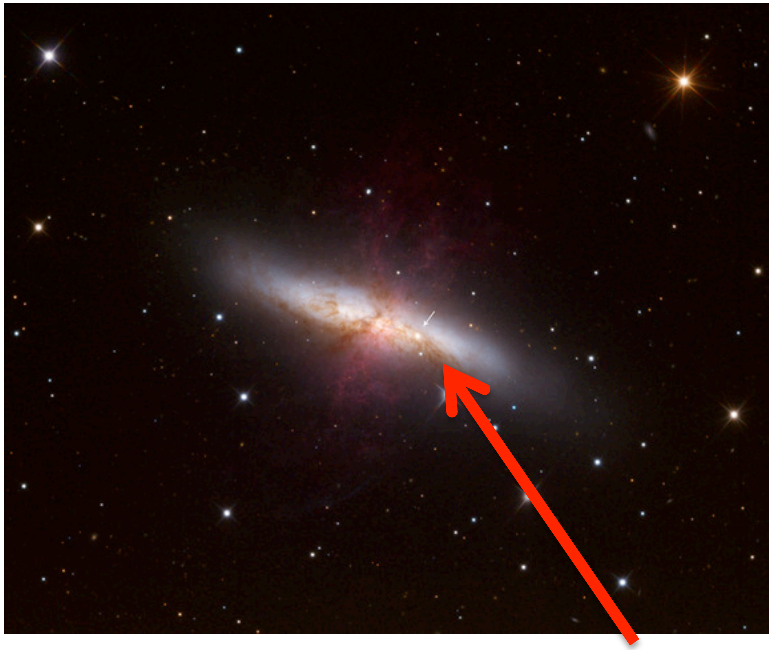
3. Observational constraints:
   neutron stars, proto-neutron stars
   core-collapse supernovae

4. Terrestrial experiments: heavy ion collisions

5. Quark-meson coupling model

6. Summary and outlook

SN 2014J is a type-Ia supernova in Messier 82 (the 'Cigar Galaxy', M82) discovered in January 2014



Concept of infinite dense matter:

System of an infinite number of interacting particles in an infinite volume with a finite ratio of a number of particles per unit volume.

No Coulomb force present – no surface effects –
- translational invariance

Practical use:
interior of neutron stars, core-collapse supernovae,
possibly large heavy nuclei

Testing theories under simplified conditions

Phases of dense matter:

Nuclear matter: symmetric (equal number of protons and neutrons)

benchmark "magic" numbers for construction of empirical models of high density matter

$$\rho_0, \ \ \mathrm{E/A}\left(\rho_0\right), \mathrm{S}(\rho_0), \ \mathrm{K}_\infty$$

Saturation density 0.16 fm^{-3}
Saturation energy 16 MeV
Symmetry energy ~ 30 MeV
Incompressibility: traditional 240+/-30 MeV
NEW VALUE 250 – 315 MeV

Asymmetric (unequal number of protons and neutrons)

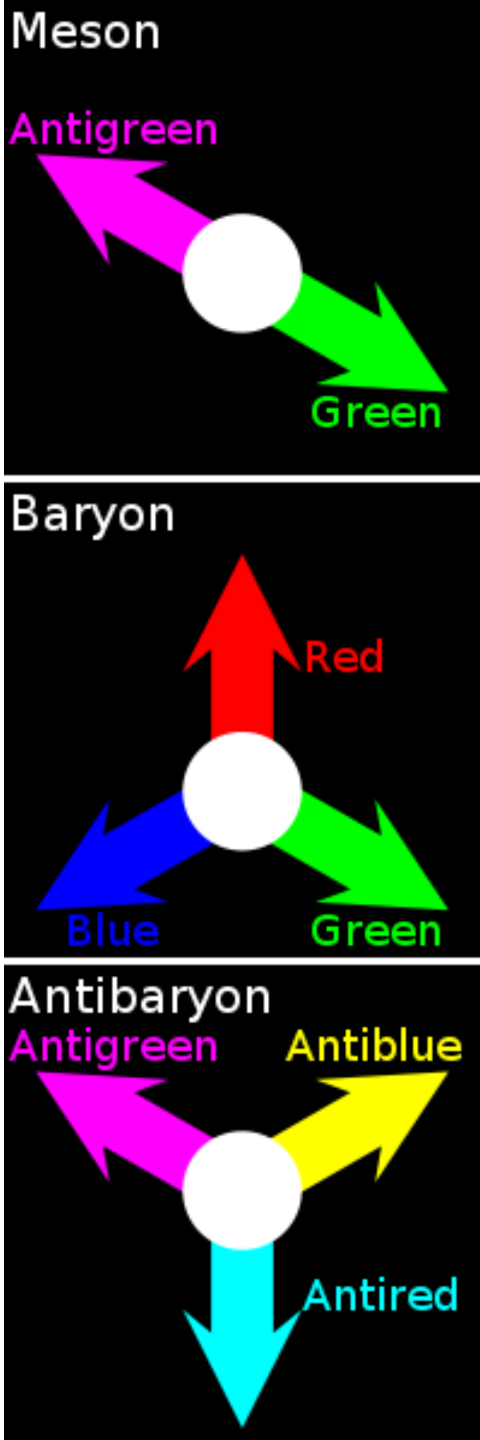
Pure neutron matter

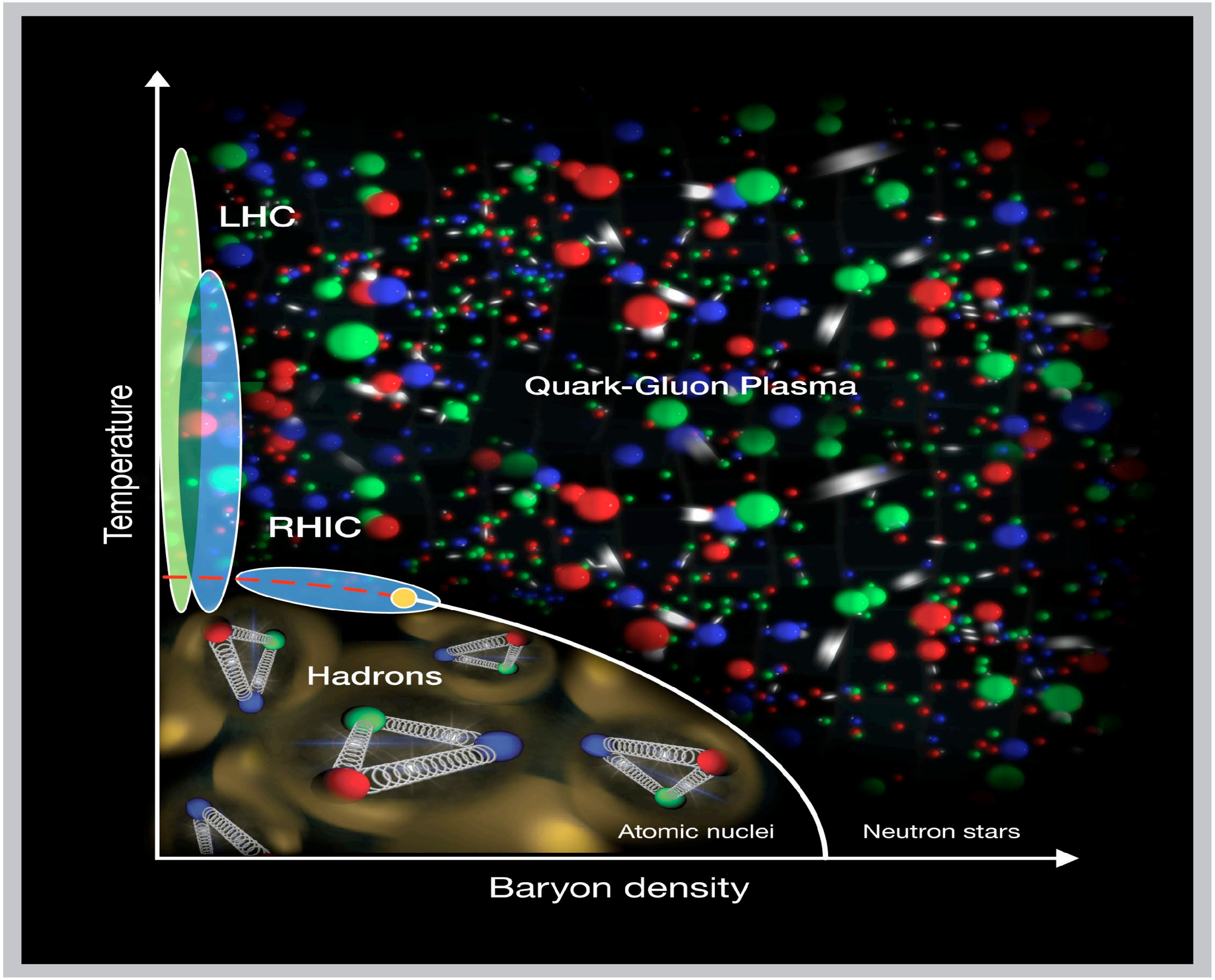
More generally:

Hadronic (objects made of quarks) **matter:**

Baryons: nucleons, hyperons
Mesons: pion and kaon condensates

Quark matter: u-d-s matter and (color)
superconducting phases





Structure of high density matter:

Starting: Collins and Perry, PRL 34, 1353 (1975)

Still open questions in 2014:

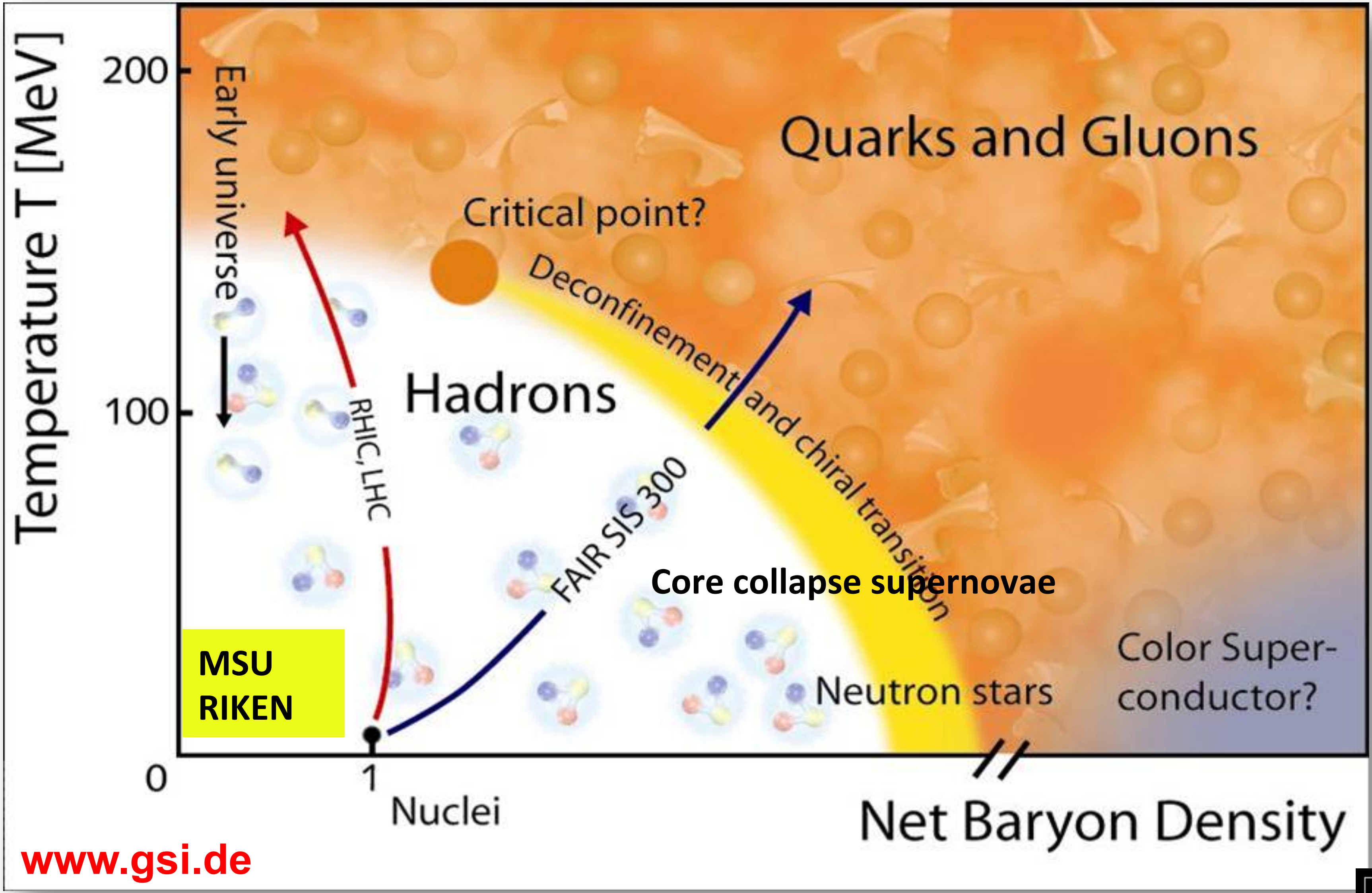
At what density baryons and mesons will start to loose their identity as bound 3(2)-quark objects?

How would this density compare to the threshold density for creating of hyperons, pions and kaons?

How to incorporate these effects into models?

How can these effects be unambiguously identified in observations?

QCD phase diagram



The Equation of State (EoS):

Ludwig Boltzmann
1844 - 1906

Ideal gas:

Average pressure :

$$p = \frac{1}{3}\frac{N}{V} m \bar{v}^2$$ N # of molecules of mass m in volume V

Average molecular kinetic energy :

$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{3}{2} \mathrm{kT}$$ k Boltzmann constant, T temperature

Equation of State

$$p = \frac{\mathrm{NkT}}{\mathrm{V}} = \varepsilon(\rho, T)$$

ε total energy density of gas with number density $\rho = \frac{N}{V}$

Nuclear matter:

$$P = \varepsilon(\rho, T) \qquad \varepsilon(\rho, T) = \sum_f \left(\frac{E}{A}(\rho, T)\, \rho \right)_f \qquad \mu_B = (P + \varepsilon)/\rho$$

Two key points:

I. The EoS is dependent on composition
CONSTITUENTS + INTERACTIONS

II E/A and ITS DENSITY DEPENDENCE
must be determined by nuclear and/or particle models.

Two key points:

The EoS is dependent on composition
CONSTITUENTS + INTERACTIONS

ε_f and ITS DENSITY AND TEMPERATURE
DEPENDENCE

must be determined by nuclear
and/or particle models.

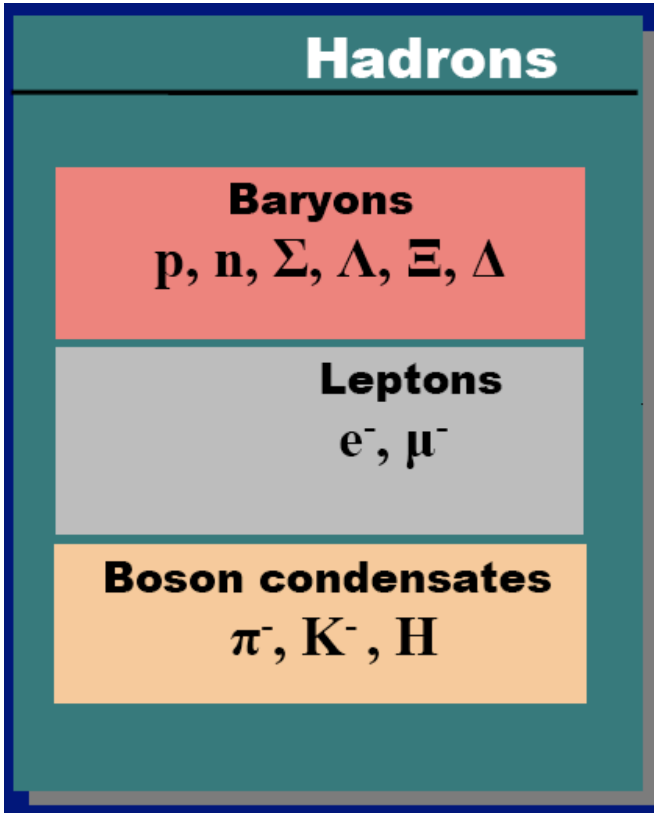
Hadronic matter:

Many variants of microscopic
and phenomenological models
at a different level of complexity:

Mean-field (non)relativistic models

"Ab initio" models
with 2- and 3-body forces

Quark-Meson-Coupling model



Hadrons

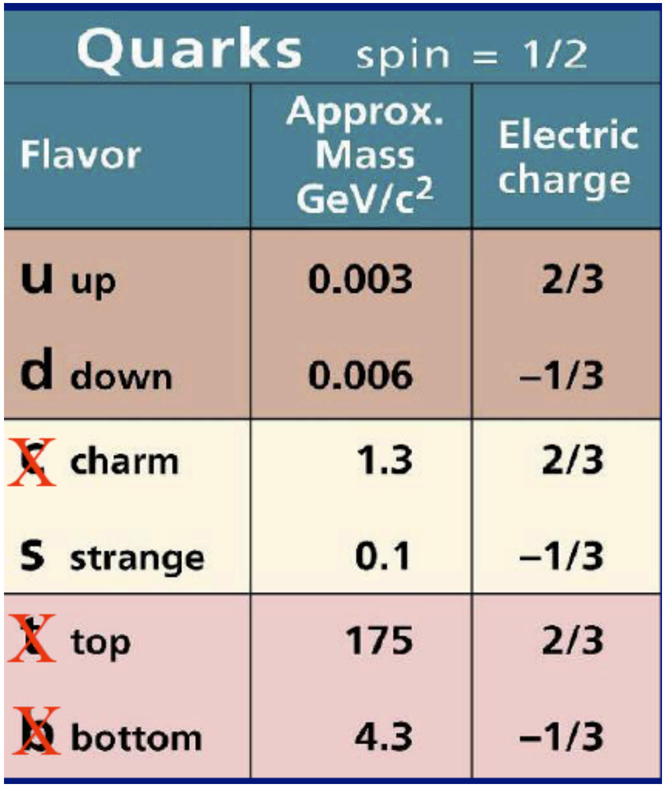
Baryons
p, n, Σ, Λ, Ξ, Δ

Leptons
e^-, μ^-

Boson condensates
π^-, K^-, H

Quark matter:

MIT bag
Nambu–Jona–Lasinio (NJL)
Polyakov – NJL (PNJL)
Polyakov–Quark Meson (PQM)
Chromo–dielectric (CDM),
Dyson–Schwinger (DS)



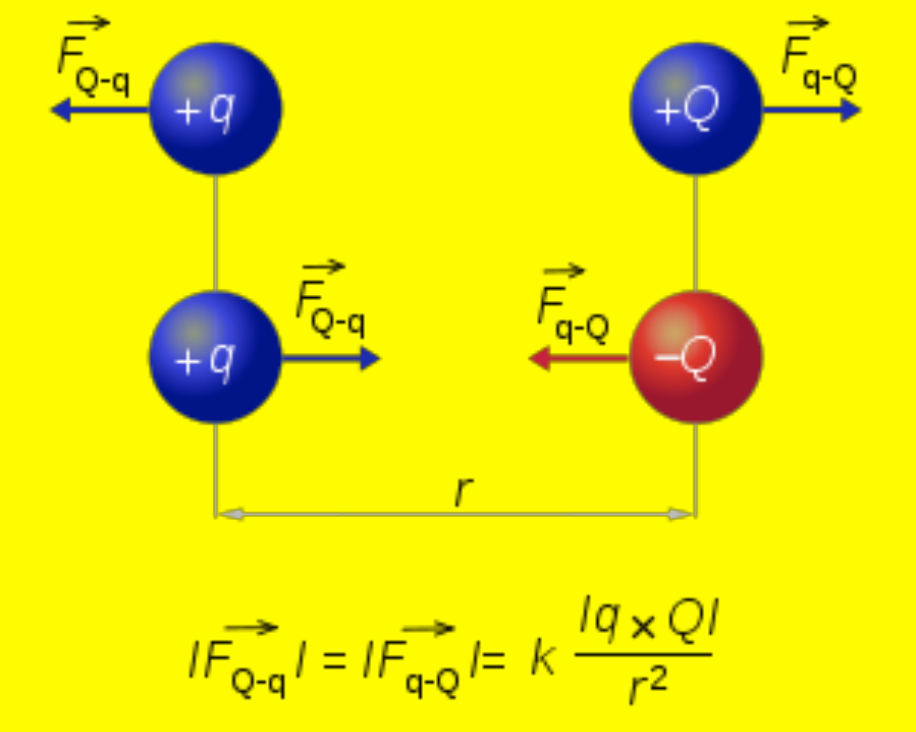
Quarks spin = 1/2

Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	–1/3
~~c~~ charm	1.3	2/3
s strange	0.1	–1/3
~~t~~ top	175	2/3
~~b~~ bottom	4.3	–1/3

Forces (interactions) between the constituents are not known. Each model HAS FREE PARAMETERS which has to fitted to data.

Coulomb force:

2 electrical charges:



$$|\vec{F}_{Q\text{-}q}| = |\vec{F}_{q\text{-}Q}| = k\,\frac{|q \times Q|}{r^2}$$

Many electrical charges:

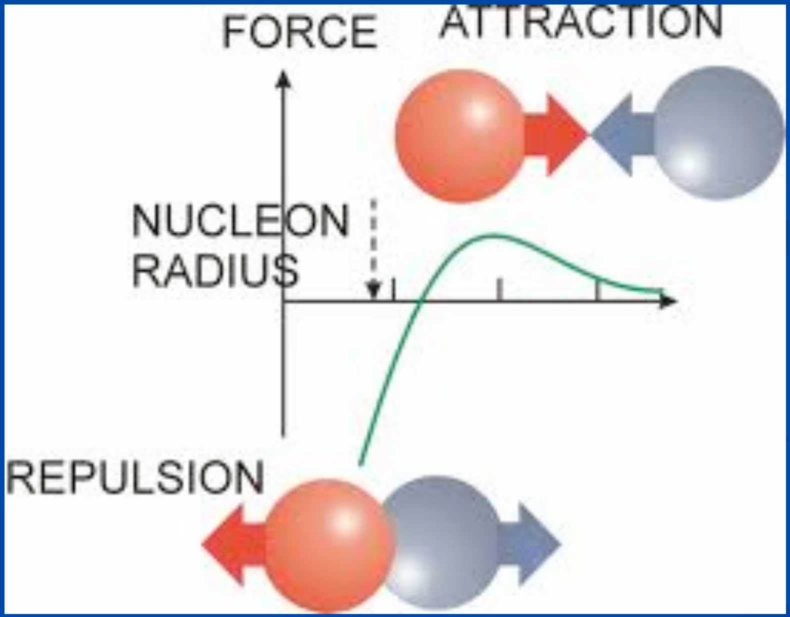
principle of superposition

Force acting on a charge q at position r due to N discrete charges:

$$F(r) = \frac{q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i (r - r_i)}{|r - r_i|^3}$$

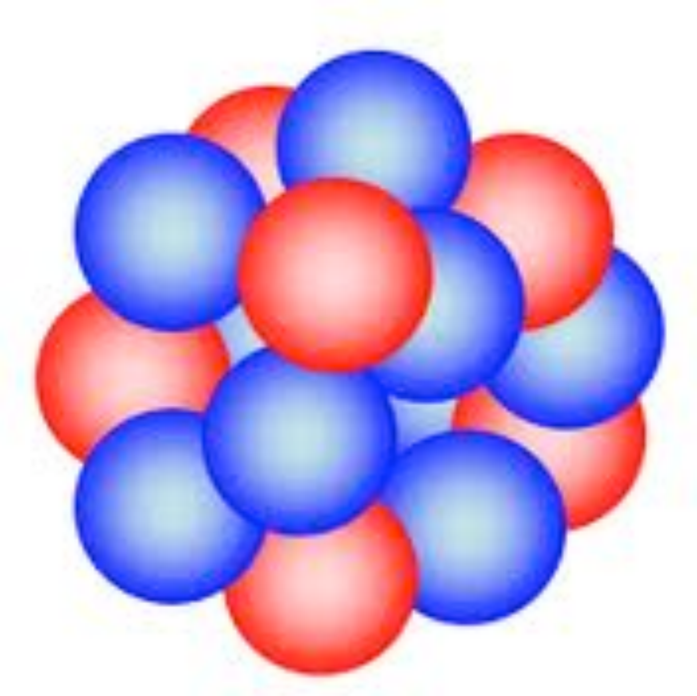
Nuclear force

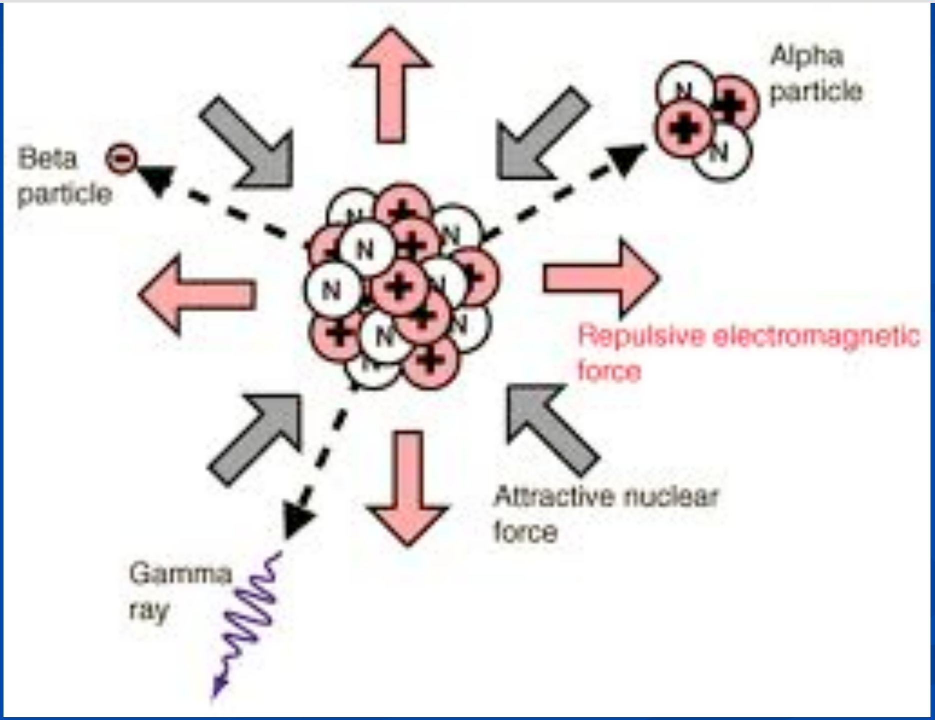
2 nucleons:
nucleon-nucleon scattering
tractable with many parameters
no unique model

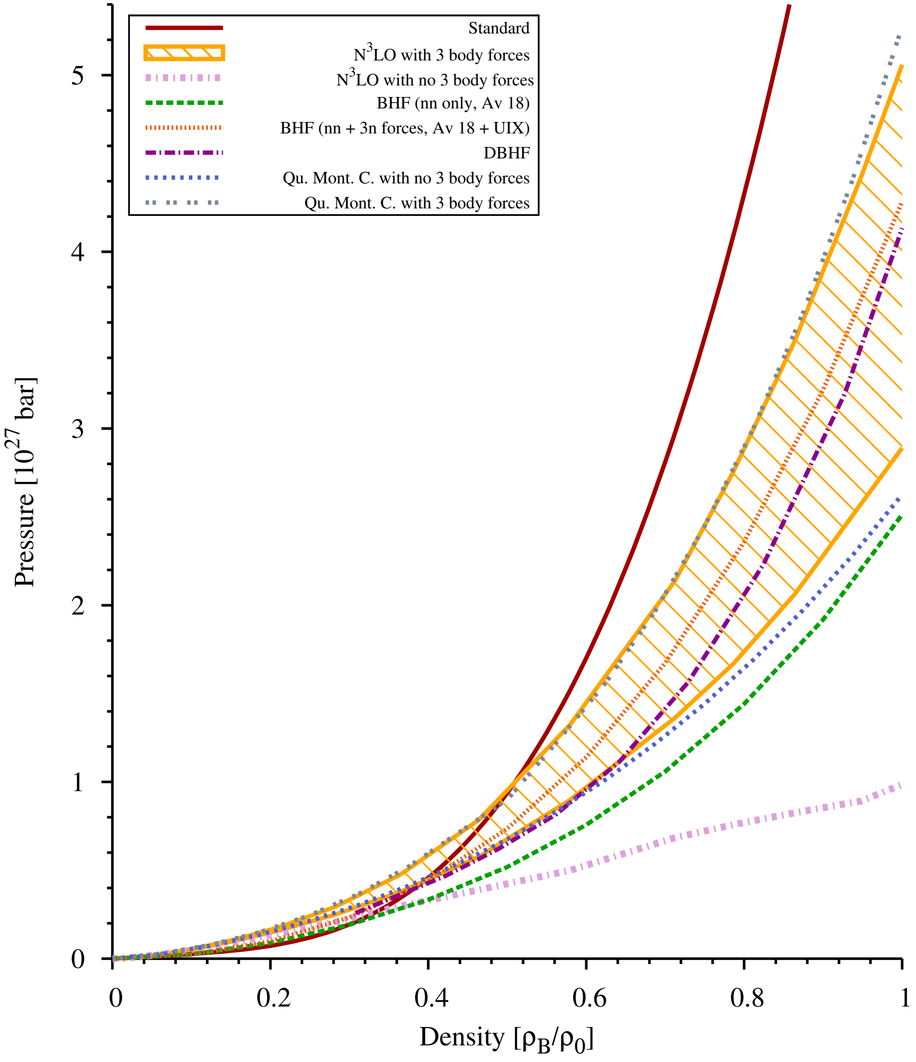


Many nucleons:

force depends on medium (density) and momentum – strong, weak and elmg interactions play role – intractable?

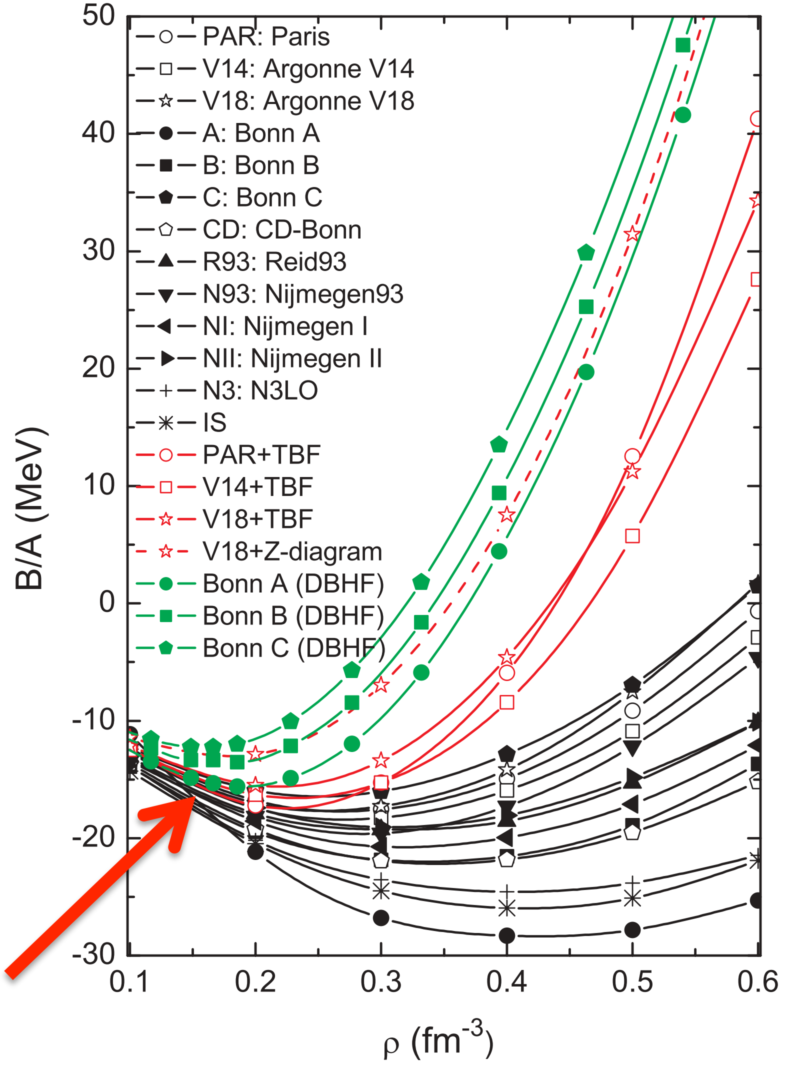






Pressure in pure neutron matter at sub-saturation density

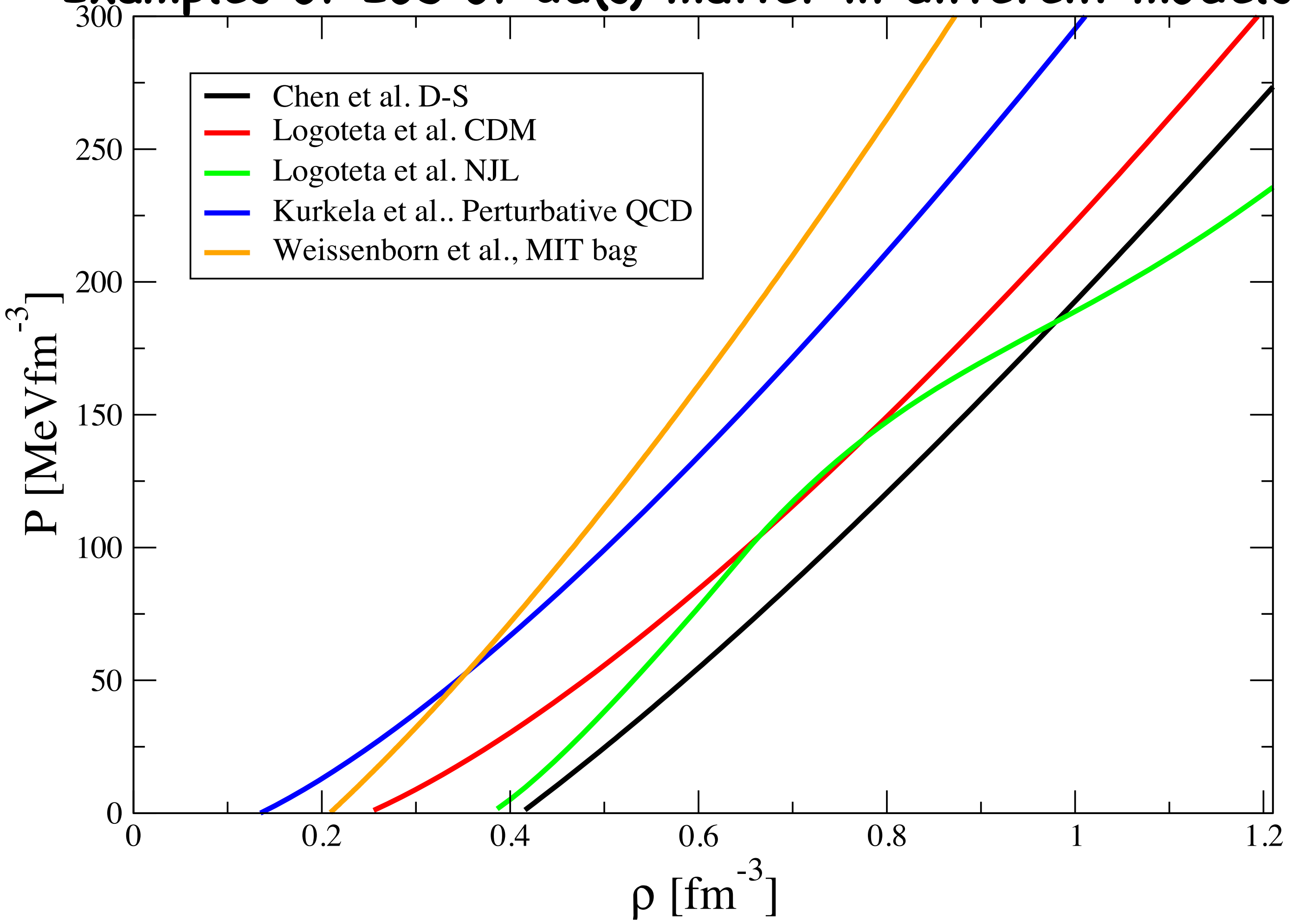
Whittenbury et al, 2013



Binding energy per particle In symmetric nuclear matter

Li et al., PRC74, 047304 (2006)

Examples of EoS of ud(s) matter in different models



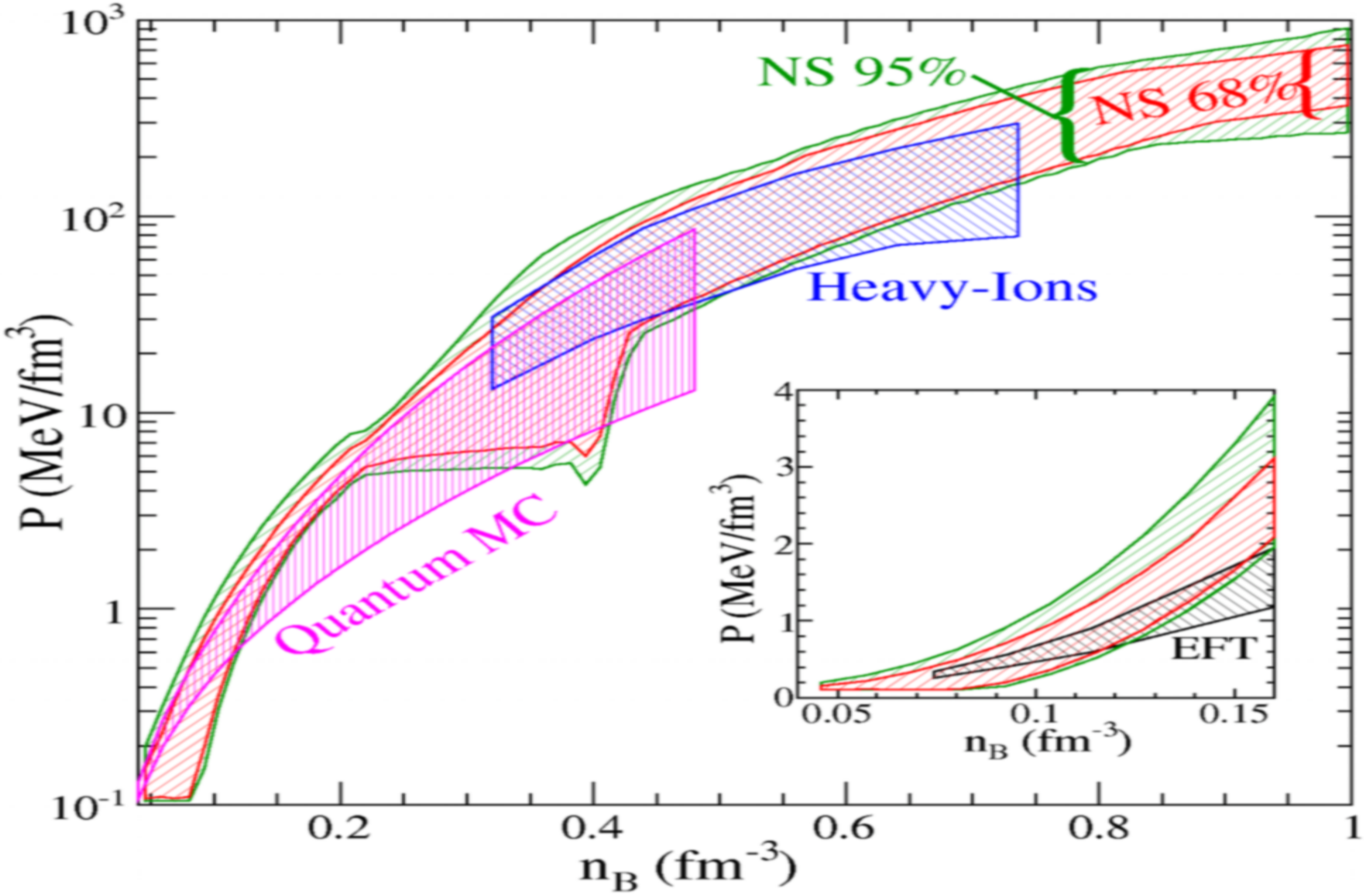
Logoteta et al., PRD85, 023003 (2012)
Chen et al., PRD86, 045006 (2012)

Kurkela et al., PRD81, 105021(2010)
Weissenborn et sl., 2011

Empirical approach:

Combination of models and observation data

Assumptions: There is only one EoS of high density matter



Steiner et al., ApJ Letters 765, L5 (2013)

Questions:

Physical content?

Predictive power?

How sensitive is observation to microphysics?

Do we have enough data to constrain our theories?

Astronomical Observation:

Neutron stars
Proto-neutron stars
Supernovae

Terrestrial Experiments:

Heavy Ion Collisions
Hypernuclei

Lattice QCD Thermodynamics:

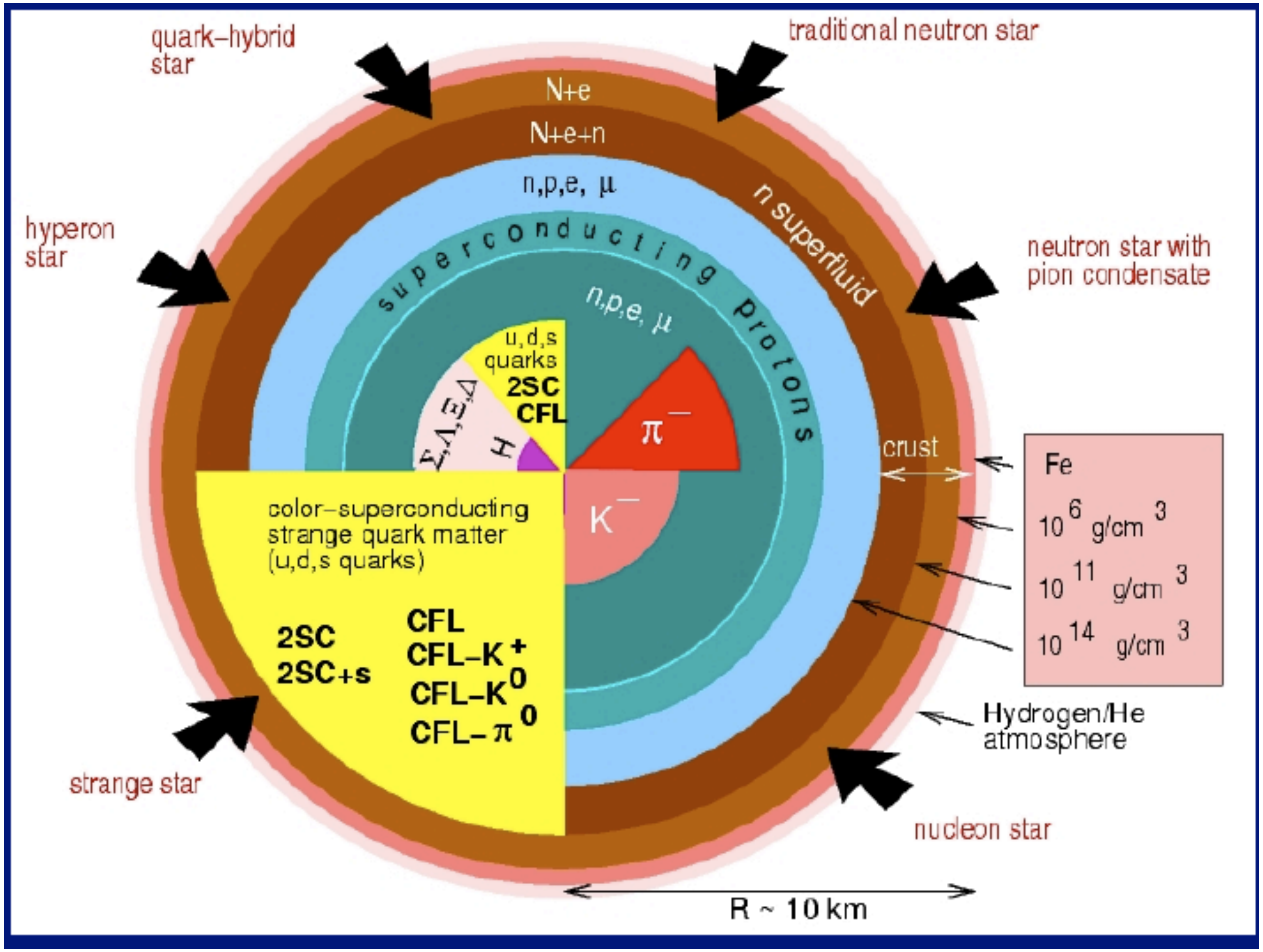
Calculation currently available only for zero baryo-chemical potential. Extrapolation to finite potential is provided by models – convergence problem.

The (T,μ) coordinates of the critical point is particularly interesting!

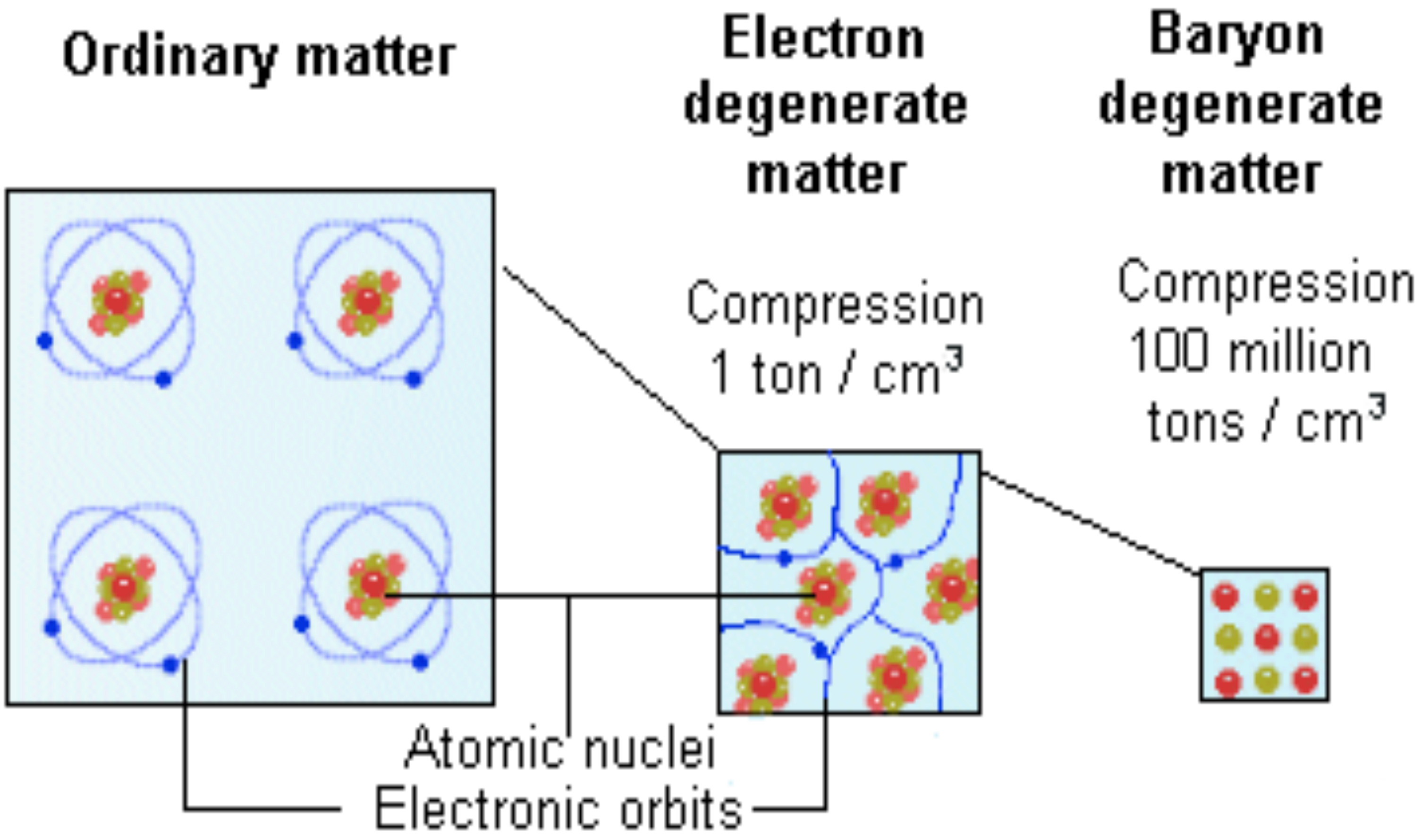
W. Weise / Progress in Particle and Nuclear Physics 67, 299-311 (2012)

Neutron Stars

Extreme conditions in neutron stars allow wide speculations about their internal structure: WHICH ARE REALLY THERE?



F. Weber Prog.Part.Nucl.Phys. 54, 193 (2005)



Ordinary matter

Electron degenerate matter

Baryon degenerate matter

Compression
1 ton / cm^3

Compression
100 million
tons / cm^3

Atomic nuclei

Electronic orbits

Basic model of (non-rotating) neutron star properties:

Tolman-Oppenheimer-Volkoff (TOV) equations for hydrostatic equilibrium of a spherical object with isotropic mass distribution in general relativity:

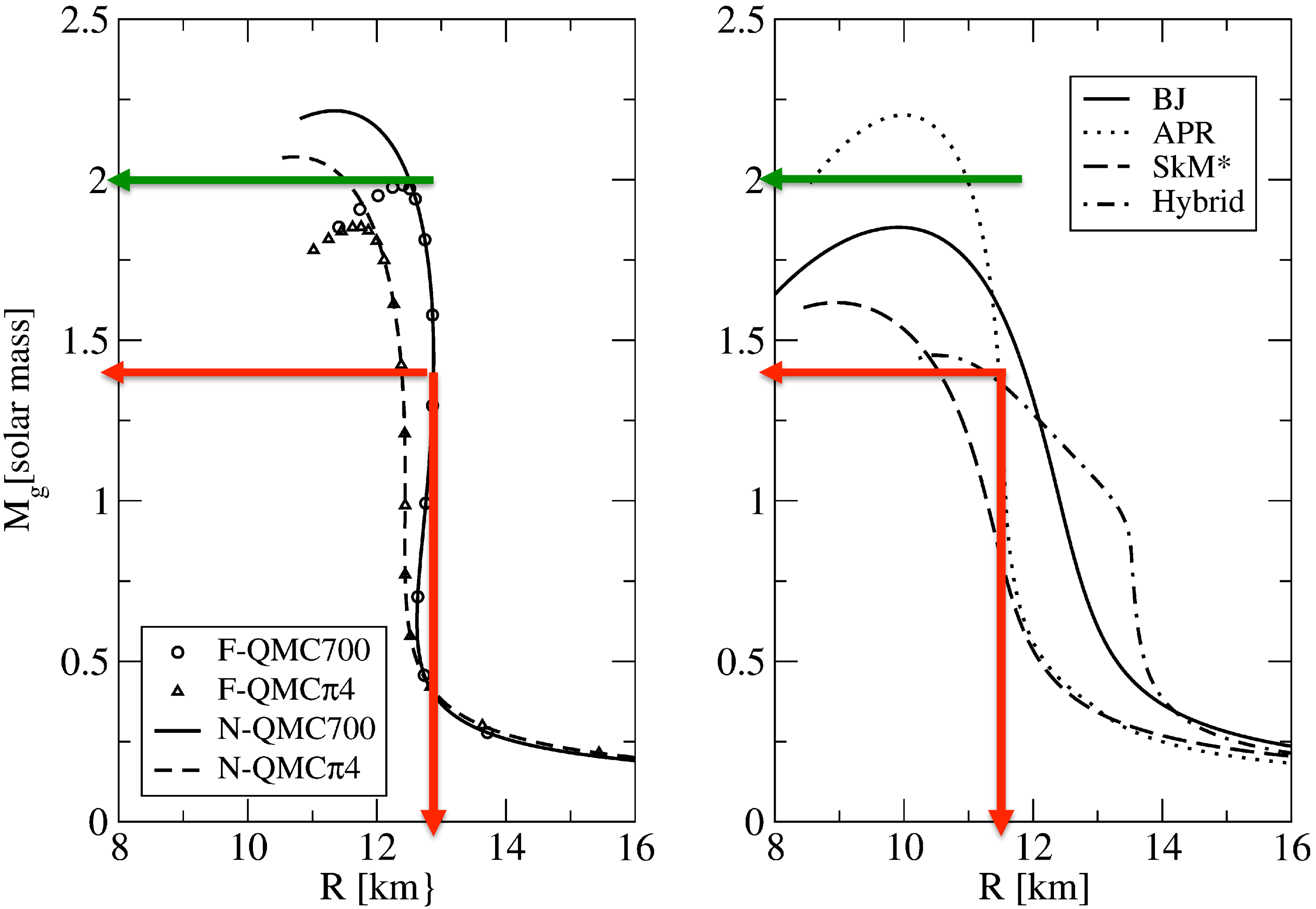
$$\frac{dP}{dr} = -\frac{GM(r)\varepsilon}{r^2}\frac{(1+P/\varepsilon c^2)(1+4\pi r^3 P/M(r)c^2)}{1-2GM(r)/rc^2}$$

$$M(r) = \int_0^r 4\pi r'^2 \varepsilon(r')dr'$$

Input: The Equation of State

P(ε) - pressure as a function of energy density

Output: Mass as a function of Radius M(R)



Is radius at maximum mass and 1.4 M⊙ a UNIQUE fingerprint of composition?

I. Precise determination of a neutron star mass is not sufficient to compare models with observation.

II. Strong dependence on the equation EoS

III. Do all observed NS have the same EoS and their M and R lie on the same M(R) curve?

A selection of five most accurately measured neutron star masses:

PSR J0737-3039 the first double pulsar (A,B)
$M = 1.249+/-0.001\ M_\odot$ (Lyne et al., Science 303, 1153 (2004))
$P = 2.77s$ (B)

PSR B1913+16 NS binary (Hulse-Taylor)
$M = 1.4414 \pm 0.0002\ M_\odot$: (Hulse and Taylor, ApJ 195, 1975)
$P = 59$ ms

PSR J1903+0327 NS on an eccentric orbit around MS star
$M=1.667 \pm 0.021\ M_\odot$: (Freire, P. C. C. et al., MNRAS, 412, 2763 (2011))
$P = 2.5$ ms

!

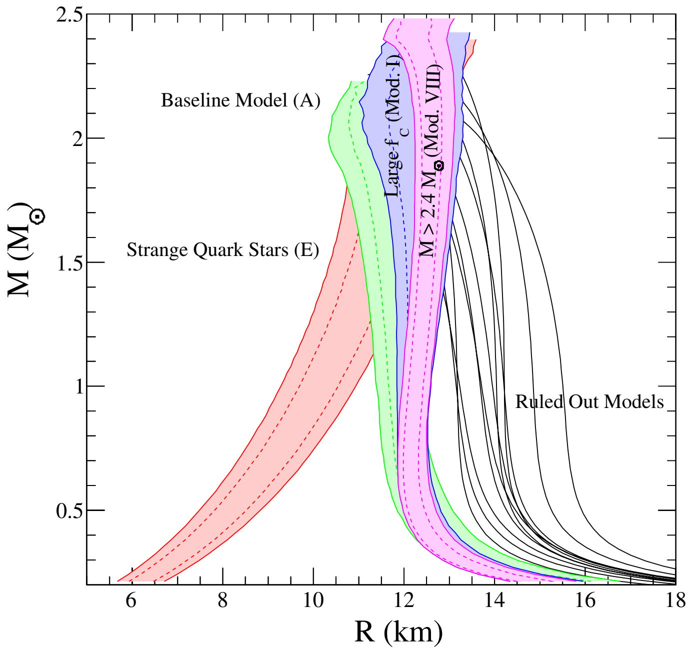
PSR J1614-2230 NS+WD
$M_g = 1.97+/-0.04\ M_\odot$ (Demorest at al., Nature 467, 1081 (2010))
$P = 3.15$ ms

!

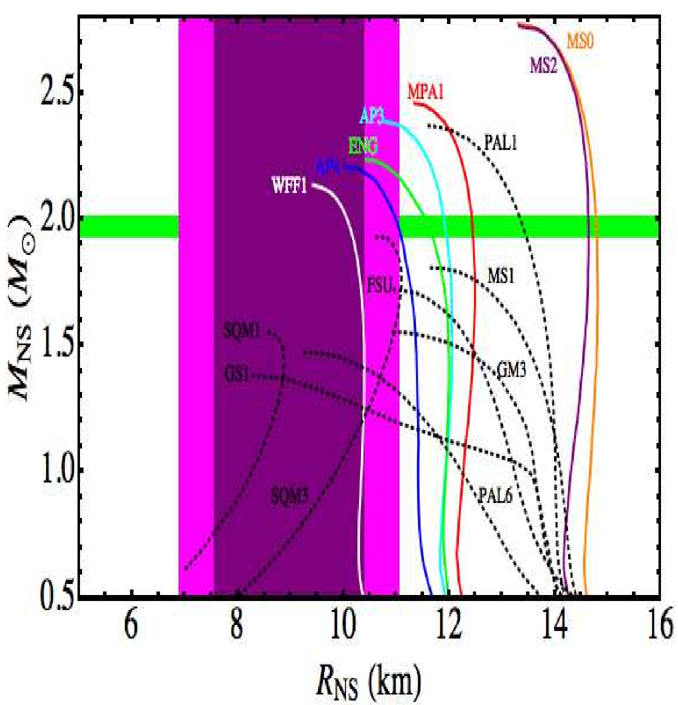
PSR J0348+0432 NS+WD
$M_g = 2.03+/-0.03\ M_\odot$ (Antoniades et al., Science 340, 448 (2013)
$P = 39$ ms

Low-mass X-ray binaries inside globular clusters (bursting and transiently accreting)

THE ASTROPHYSICAL JOURNAL LETTERS, 765:L5 (5pp), 2013 March 1



The Radius of Neutron Stars



Steiner, Lattimer and Brown:

1.4 M⊙ 10.4 – 12.9 km 90% conf

Sebastien Guillot et aL:arXiv:1302.0023

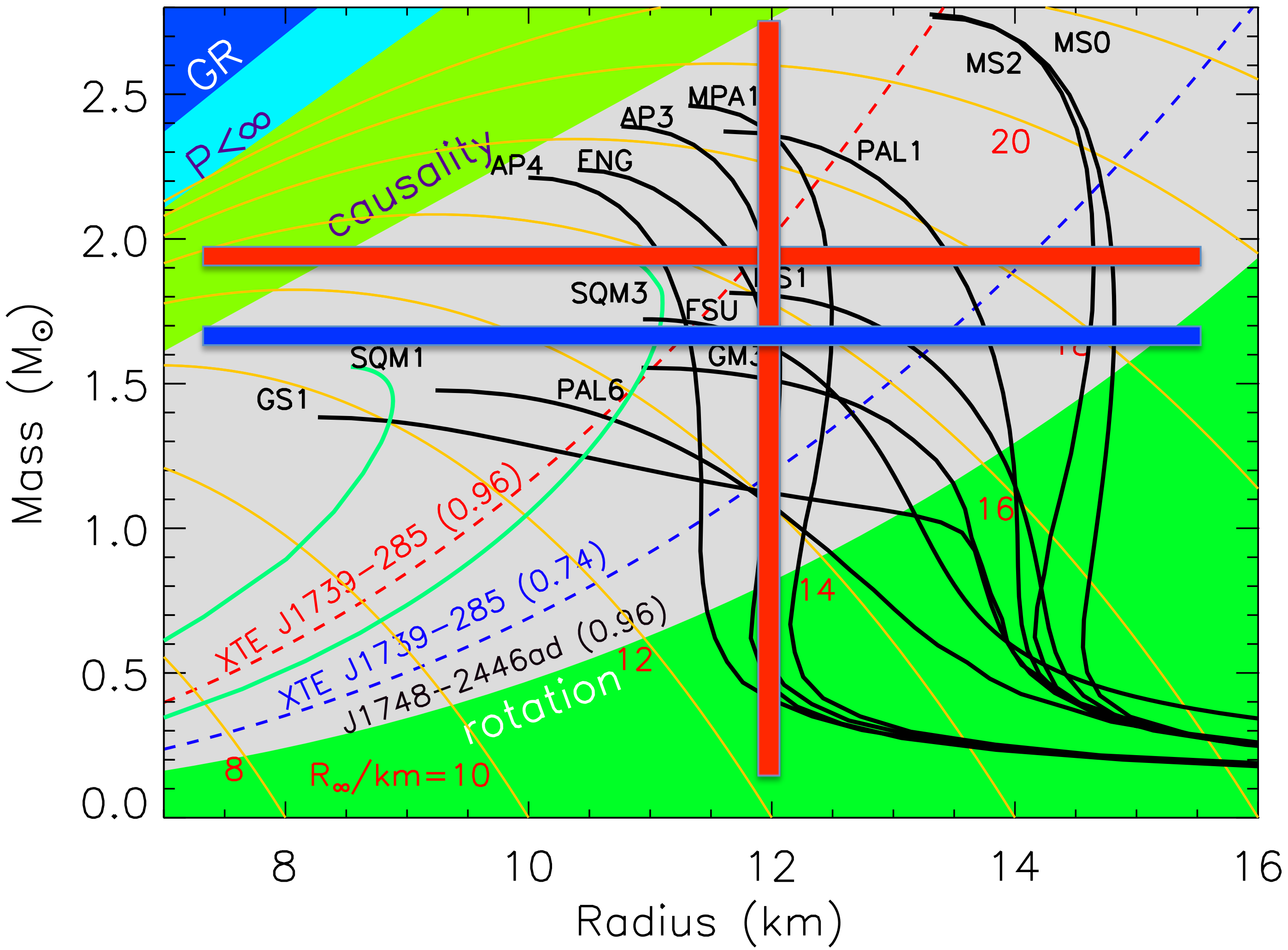
$$R_{\mathrm{NS}} = 9.1^{+1.3}_{-1.5}\,\mathrm{km}\ (90\%\text{-confidence})$$

ALL MASSES

arXiv:1305.3242 (May 2013) 90% conf

1.4 M⊙	11.4 - 12.8 km
1.2 – 2.0 M⊙	10.9 - 12.7 km

Figure from Lattimer Astrophys Space Sci (2011) 336:67-74

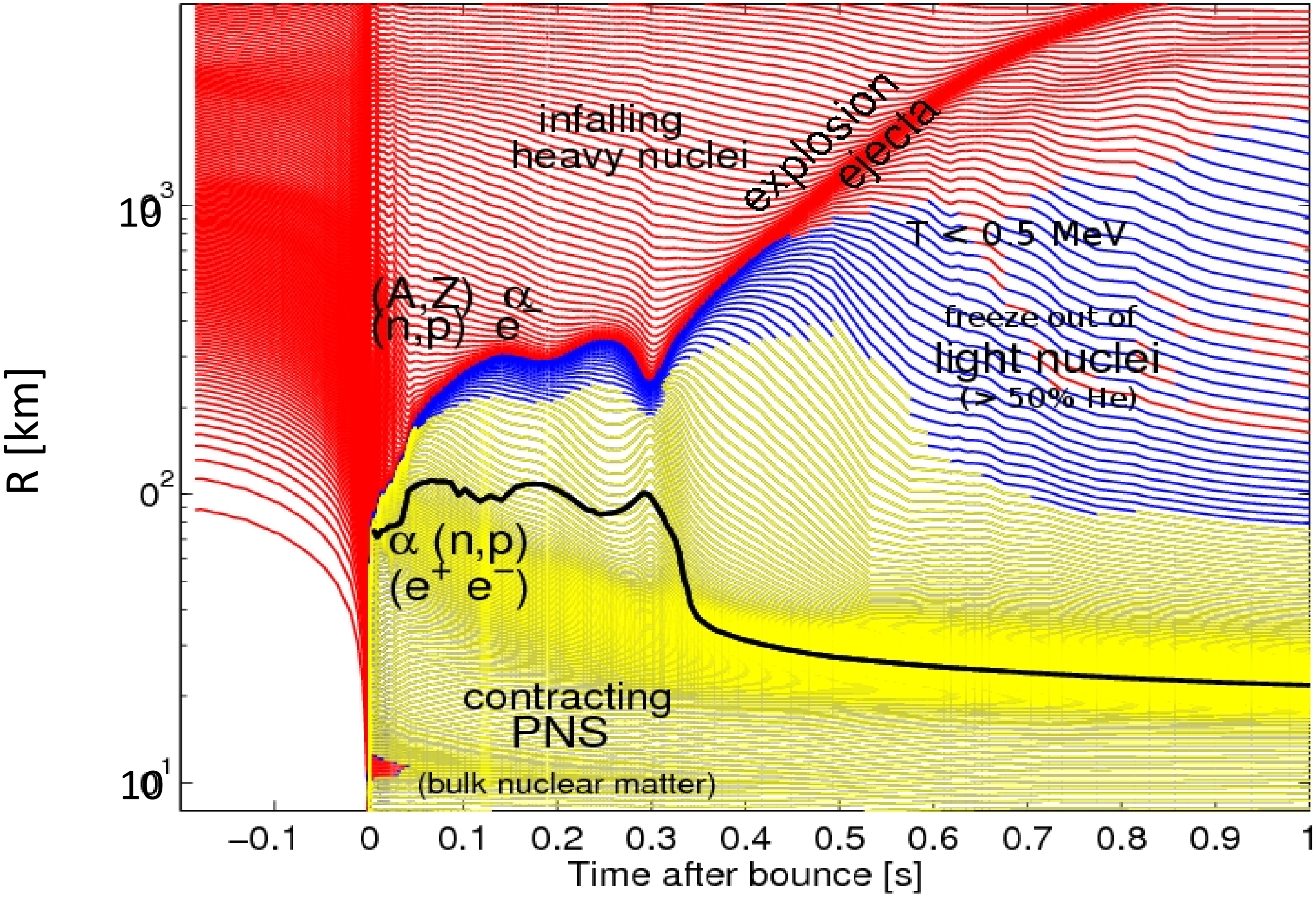


Some EoS used for calculation of gravitational masses and radii of cold neutron stars (selection by Lattimer+Prakash)

Even very precise information on mass and radius on the same object Will not fully solve the uncertainty in the EoS of neutron star matter

Proto-neutron stars and their evolution

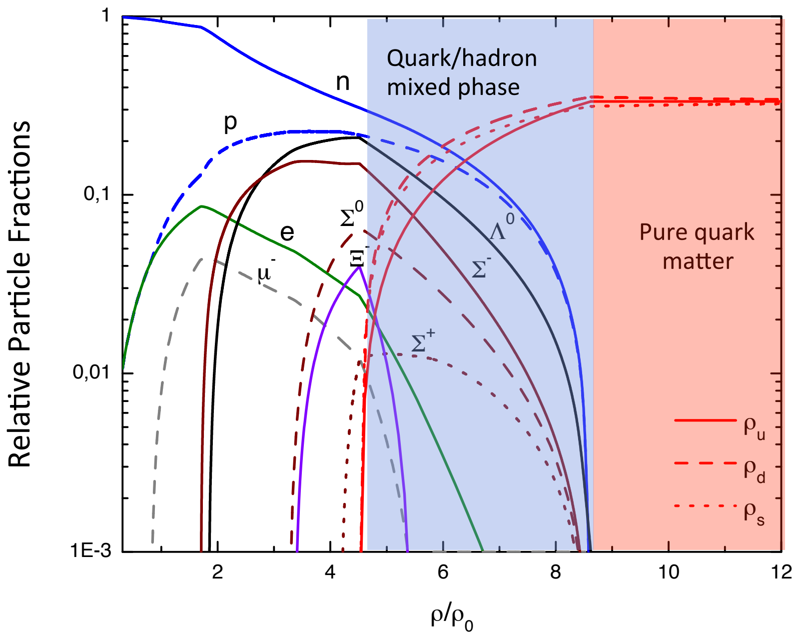
What energy density is available during the formation of the PNS? (essential time up to 60 sec after bounce)



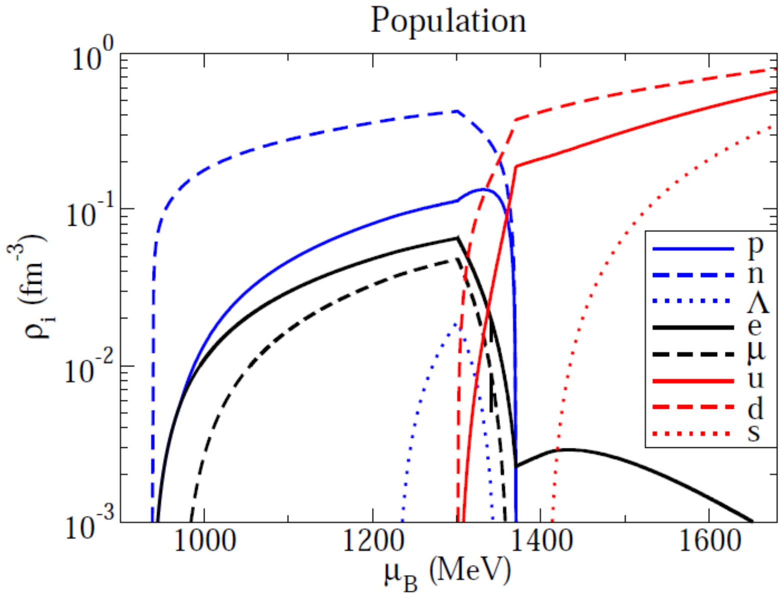
T. Fischer, talk at CSQCD II, May 2009

Model Neutron Star Matter Composition

Non-local SU(3) NJL with vector coupling



Milva Orsaria & Hilario Rodrigues (June 2012)



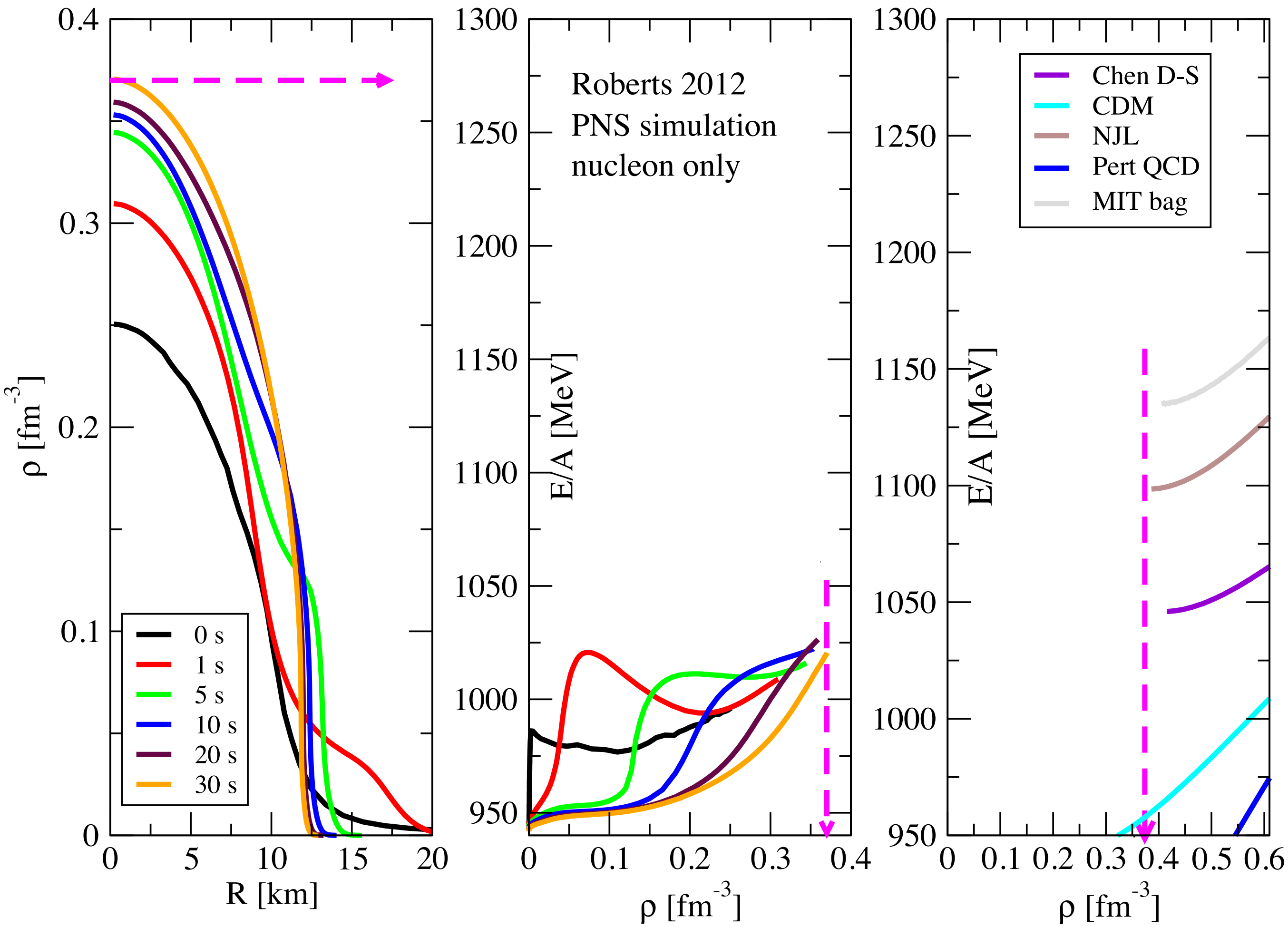
Dexheimer and Schramm, PRC81 045201 (2010)

Physical conditions for appearance: hyperons,

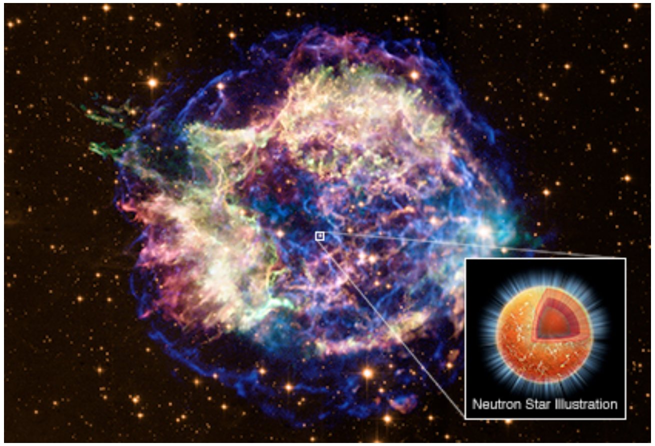
π and K meson condensates

u d s matter +

THRESHOLD DENSITIES UNKNOWN – STRONGLY MODEL DEPENDENT

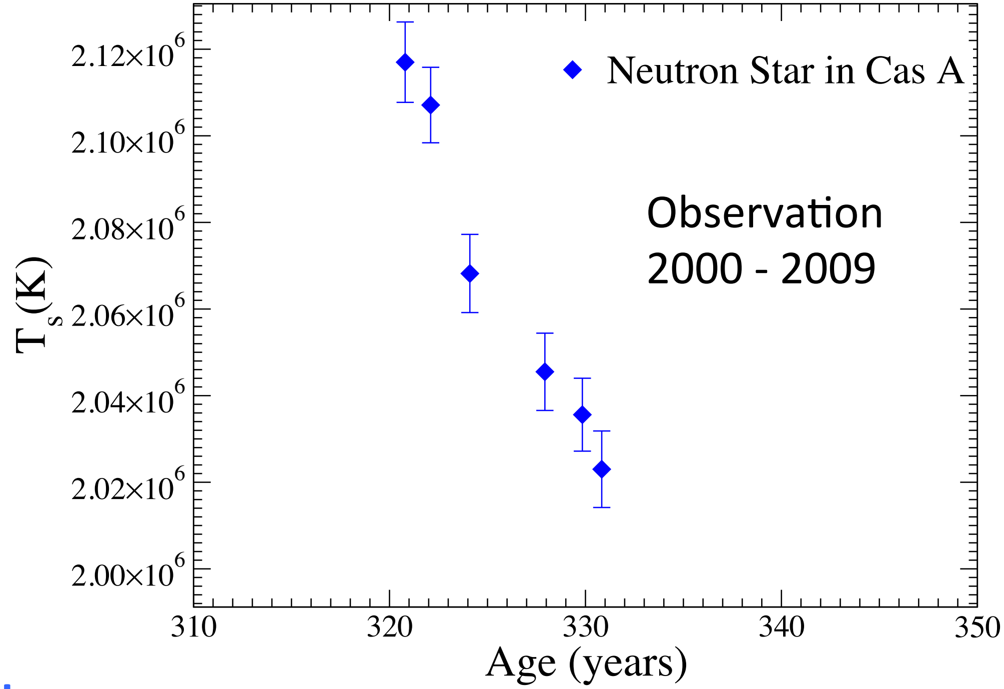


Can quarks matter be created in NS cores?



Cassiopeia A

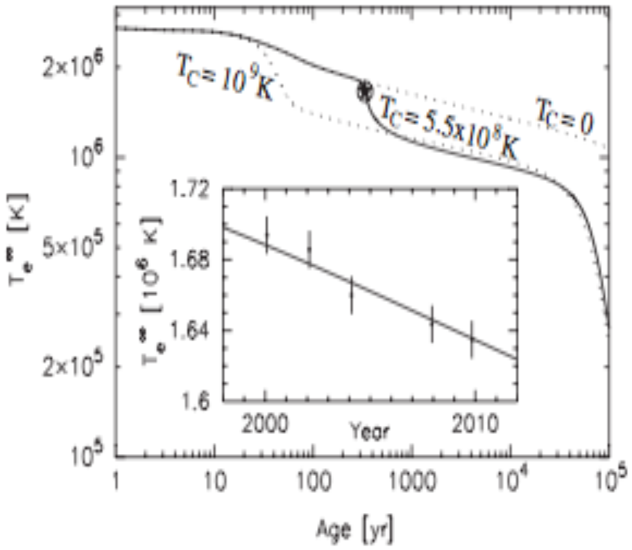
Remnant of the historical 1680 SN explosion discovered in 1999 with Chandra X-ray Observatory

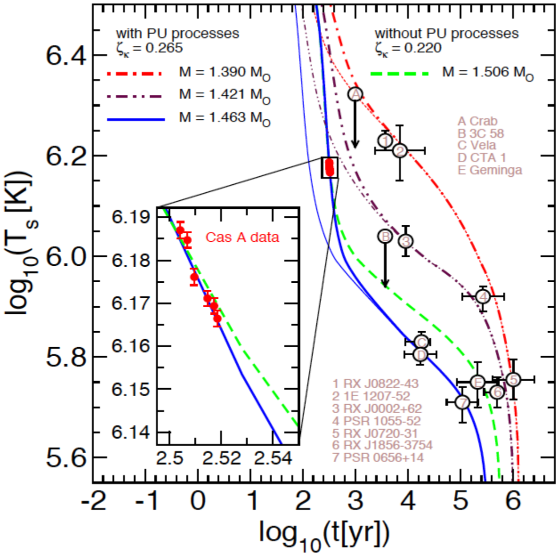


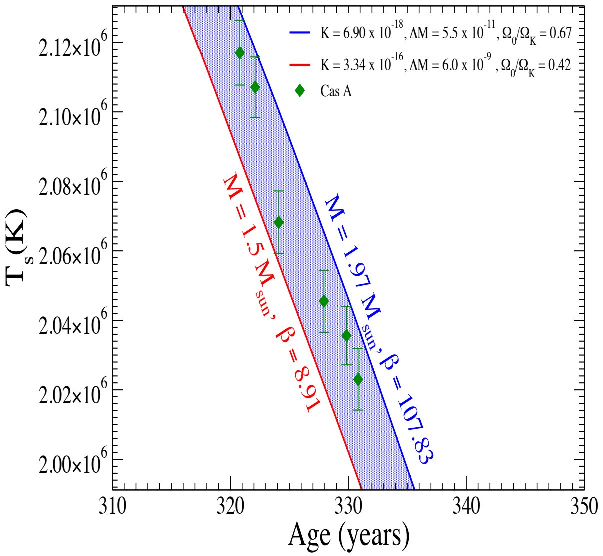
Isolated neutron star with a carbon atmosphere and low magnetic field

Precise data on rapid cooling

C. O. Heinke & W. C. G. Ho, ApJ Letters 719 (2010) L167







Rapid cooling is triggered by neutron superfluidity in dense matter enhanced by neutrino emission from the recent onset of the breaking and formation of neutron Cooper pairs in the $^3P_2$ channel in the star's core. Large proton superconductivity need to be present in the core.

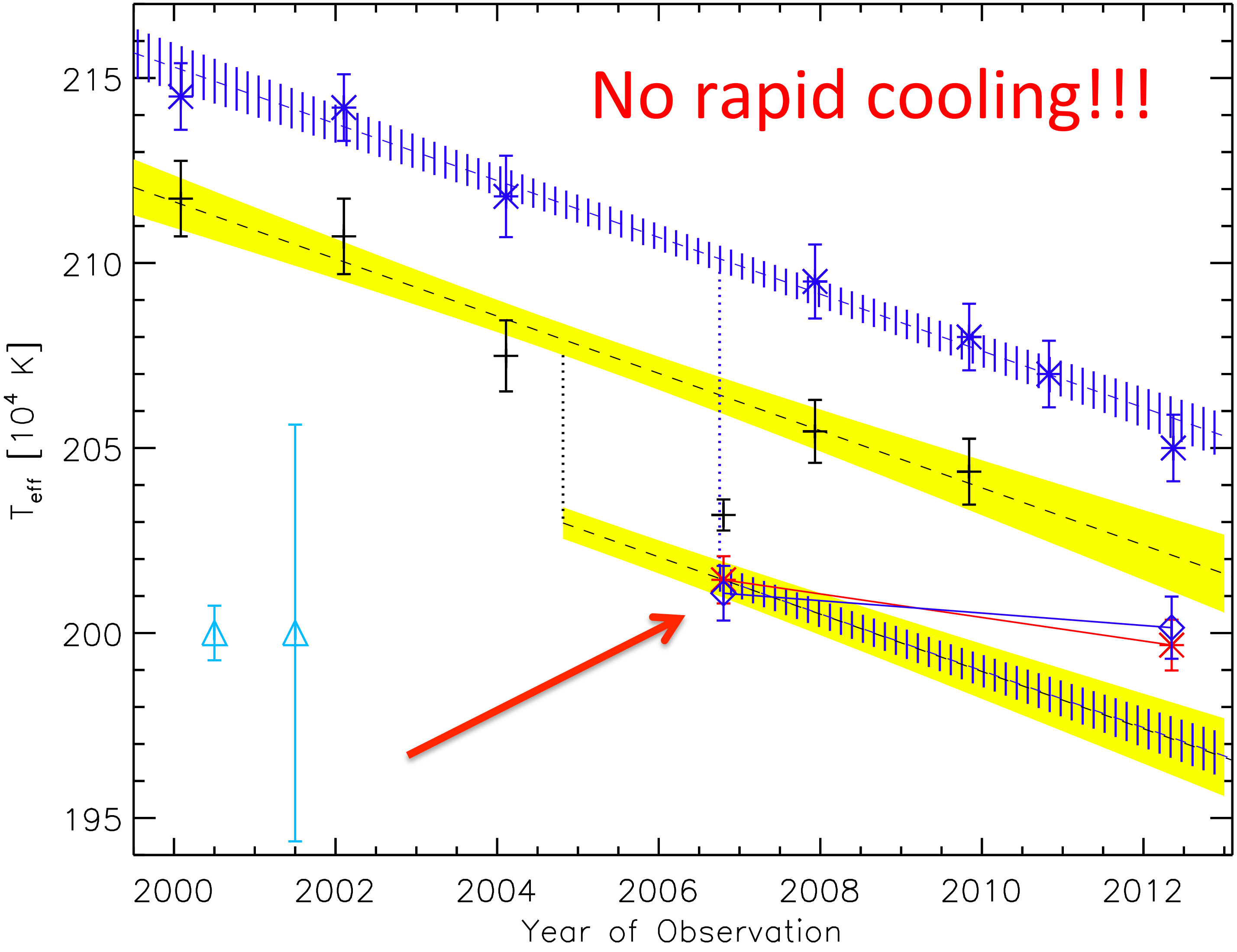
Page et al., PRL 106,081101 (2011)

The cooling rates account for medium-modified one-pion exchange in dense matter and polarization effects in the pair-breaking formations of superfluid neutrons and protons.

Blaschke et al.,
PRC 85, 022802(R) 2012

Relativistic model of a 2D rotating neutron star combined with relativistic thermal energy transport: Frequency dependent composition and temperature distribution

Weber, Compstar Tahiti 2012
Negreiros et al.,
PRD 85, 014019 (2012)



No rapid cooling!!!

THE ASTROPHYSICAL JOURNAL, 779:186 (18pp), 2013 December 20

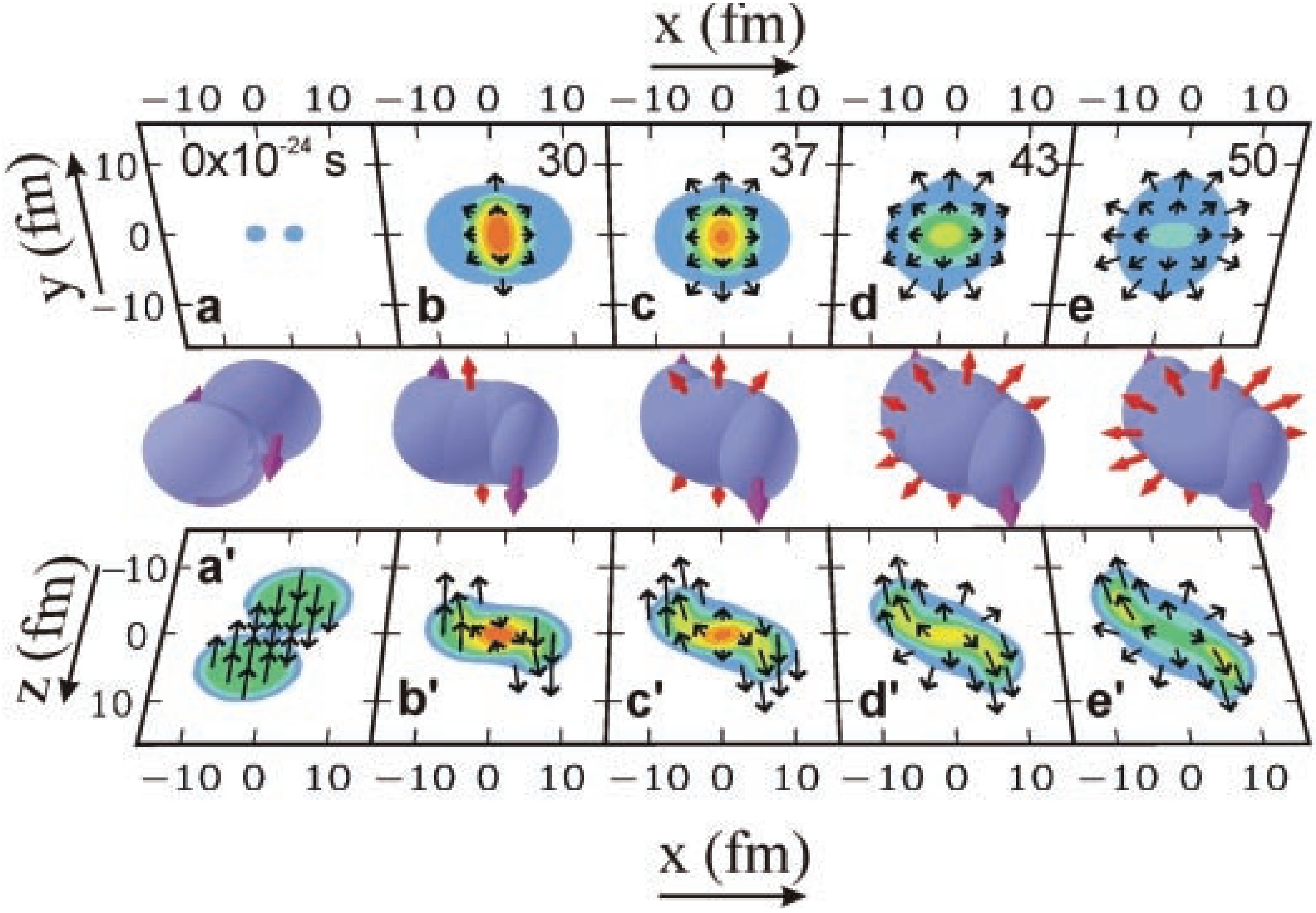
Heavy ion collisions

Heavy Ion collisions:

GSI, MSU, Texas A&M, RHIC, LHC existing
FAIR (GSI), NICA (Dubna, Russia) planned

Measurement: Beam energy 35 A MeV – 5.5 A TeV
Collisions (Au,Au), (Sn,Sn) , (Cu,Cu)
but also (p,p) for a comparison
Transverse and Elliptical particle flow

Calculation: Transport models -- empirical mean field potentials
Fit to data → energy density → P (ε) → the EoS
(extrapolation to equilibrium, zero temperature, infinite matter)
(e.g Danielewicz et al., Science 298, 2002, Bao-An Li et al., Phys.Rep. 464, 2008)

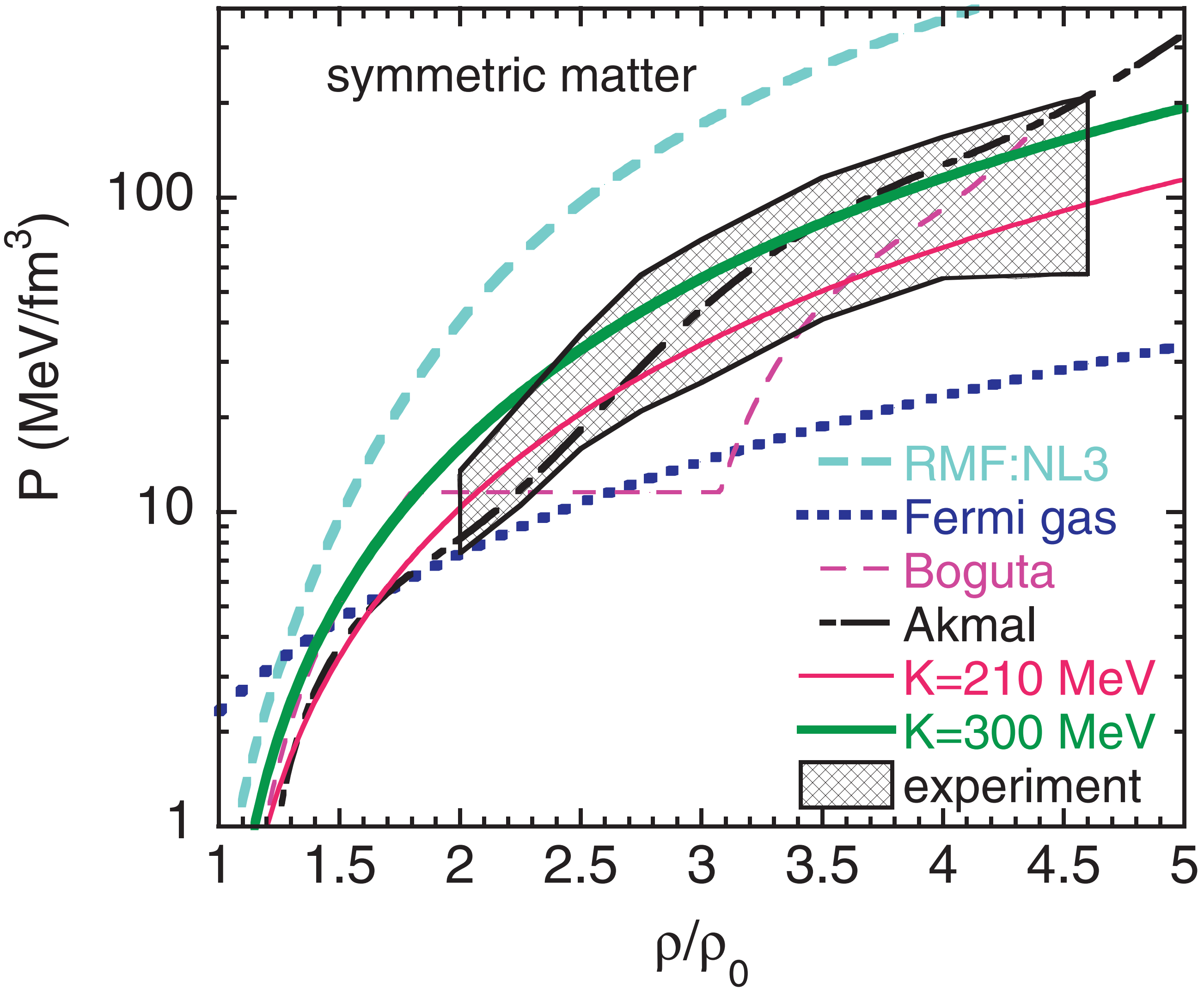


Comparison of in-plane and out-of-plane - elliptical flow
Side-ways deflection – transverse flow

Determination of the Equation of State of Dense Matter

Paweł Danielewicz,[1,2] Roy Lacey,[3] William G. Lynch[1*]

Science 298, 1592 (2002)



Transport models with parameters fitted to data on elliptical and transverse flow

MSU, Texas A&M, RIKEN 10-50 MeV/A

Central A-A collision:

Strongly beam energy dependent
Beam energy < 1GeV/ A:

Temperature: < 50 MeV
Energy density: ~ 1 -2 GeV/fm^3
Baryon density < ρ_0
Time scale to cool-down: 10^{-22-24} s
No neutrinos

Strong Interaction: (S, B and L conserved)
Time scale 10^{-24} s

Inelastic NN scatterings,
N,N*, Δ's
LOTS of PIONS
strangeness
less important (kaons)

? EQUILIBRIUM?

Proto-neutron star:

(progenitor mass dependent)
~ 8 – 20 solar mass

Temperature: < 50 MeV
Energy density: ~ 1 GeV/fm3
Baryon density ~ 2-3 ρ_s
Time scale to cool-down: 1 -10 s
Neutrino rich matter

Strong +Weak Interaction: (B and L con)
Time scale 10^{-10} s

Higher T: strangeness produced in
in weak processes
Lower T: freeze-out

N, strange baryons and mesons,
NO PIONS, leptons

?EQUILIBRIUM?

Observation and experiment does not allow to constrain current theoretical models of high density matter

Similar situation in low energy nuclear structure

Try models with parameters constrained by basic physical priciples

QUARK-MESON-COUPLING MODEL

History:

Original: **Pierre Guichon (Saclay), Tony Thomas (Adelaide) 1980'**

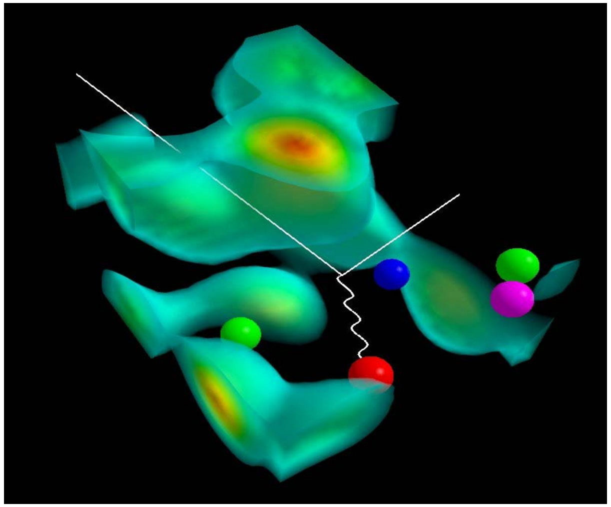
Several variants developed in Japan, Europe, Brazil, Korea, China

Latest: **Whittenbury et al. arXiv:1307.4166v1, July 2013**

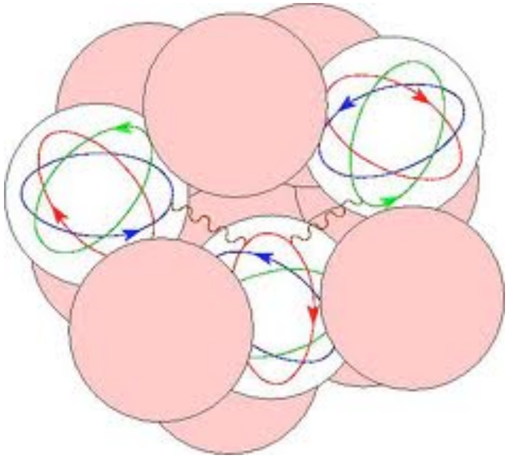
Main idea:

Effective model of the MEDIUM EFFECT on baryon structure and interactions

Quark level – coupling between u and d quarks of non-overlapping baryons by meson exchange - significantly simplifies as compared to nucleonic level.



QCD inspired *(Thomas)*



Schematic *(Guichon)*

WHAT WE DO:

1. Take a baryon in medium as an MIT bag (with one qluon exchange) immersed in a mean scalar field (NJL in progress)

2. Self-consistently include the effects of local couplings of the u and d quarks to a scalar-isoscalar meson (σ) mean field, generated by all the other hadrons in the medium, on the internal structure of that hadron.

3. Calculate the effective mass of the baryon

$$M_B^* = M_B - w_{\sigma B} g_{\sigma N} \bar{\sigma} + \frac{d}{2} \tilde{w}_{\sigma B} \left(g_{\sigma N} \bar{\sigma} \right)^2$$

where $g_{\sigma N}$ are CALCULATED coupling constants and $w_{\sigma B}$ are weighting factors allowing using unique σ-N coupling for other baryons. The modification of the internal baryon structure is the only place the quark degrees of freedom enter the model.

4. **Construct QMC Lagrangian on a hadronic level in the same way as in RMF but using the effective baryon mass M^*_B. and proceed to calculate standard observables.**

5. **Technically: Full (exchange) Fock term is included (vector and tensor), and σωρπ mesons**

(*For technical details see Whittenbury et al. arXiv:1307.4166v1*

Parameters (very little maneuvering space) :

meson-quark coupling constants:

$$g^q_\sigma,\ g^q_\omega,\ \text{and}\ g^q_\rho\ \text{for}\ q = u, d\ (g^s_\alpha = 0\ \text{for all mesons}\ \alpha).$$

Fixed to saturation density 0.16 fm^{-3}, binding energy of SNM -16 MeV and the symmetry energy 32.5 MeV

Meson masses: ω, ρ, π keep their physical values
σ = 700 MeV

Cut-off parameter Λ (in form-factors in the exchange terms)
constrained between 0.9 and 1.3 GeV

Free nucleon radius: 1 fm (limited sensitivity within change +/- 20%)

All other parameters either calculated or fixed by symmetry.

WHAT WE GET:

1. Model formulated on quark level which can tackle fundamental issues of nuclear structure within QCD that cannot be addressed by low-energy nuclear theory alone.

2. Scalar polarizability of the baryon:

$$M_B^* = M_B - g_{\sigma B}\sigma + \frac{d}{2}\left(g_{\sigma B}\sigma\right)^2$$

Atoms: re-arrangement to oppose the effect of external field – polarization

Nucleons: self-consistent response to the applied mean scalar field tends to oppose that applied field.
Increase in the scalar field effectively decreases coupling of the σ to an in-medium baryon → the baryons are source of of the scalar field → saturation (equilibrium) will be reached.

NATURAL EXPLANATION FOR SATURATION OF NUCLEAR MATTER

Hyperons

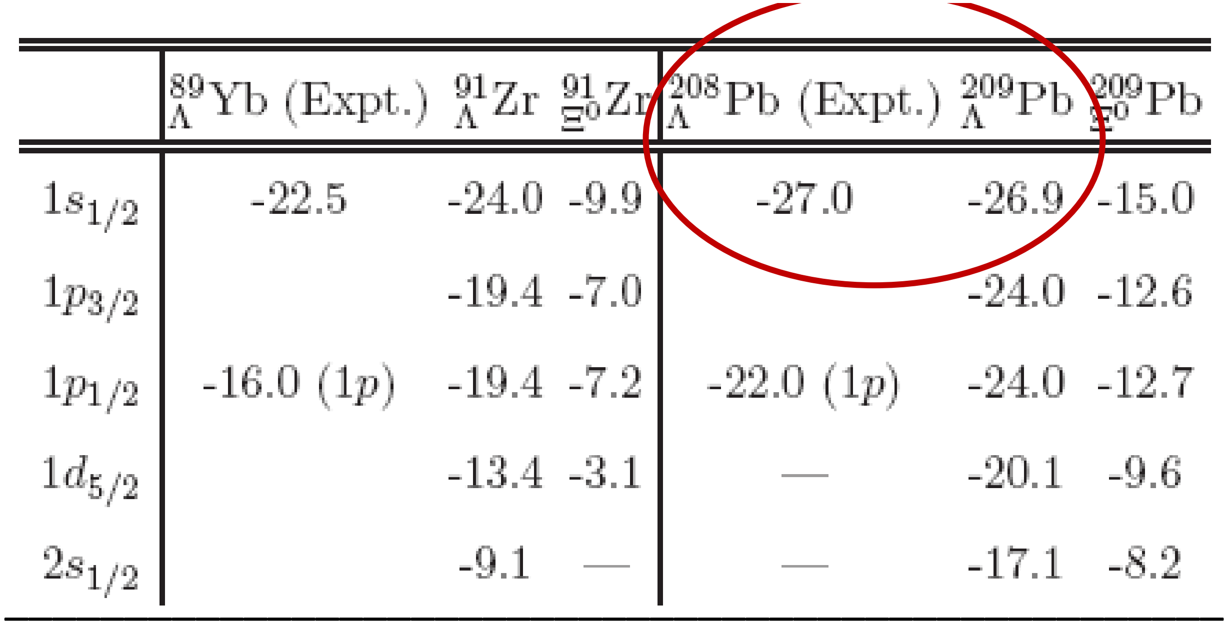
P. A. M. Guichon, A. W. Thomas and K. Tsushima, Nucl. Phys. A 814, 66 (2008).

- Derive Λ N, Σ N, Λ Λ effective forces in-medium with **no** additional free parameters
- Attractive and repulsive forces (σ and ω mean fields) both decrease as # light quarks decreases
- NO Σ hypernuclei are bound!
- Λ bound by about 30 MeV in nuclear matter (~Pb)
- Nothing known about Ξ hypernuclei – JPARC!

Λ and Ξ hypernuclei in QMC:

P. A. M. Guichon, A. W. Thomas and K. Tsushima, Nucl. Phys. A 814, 66 (2008).

Calculation without additional parameters

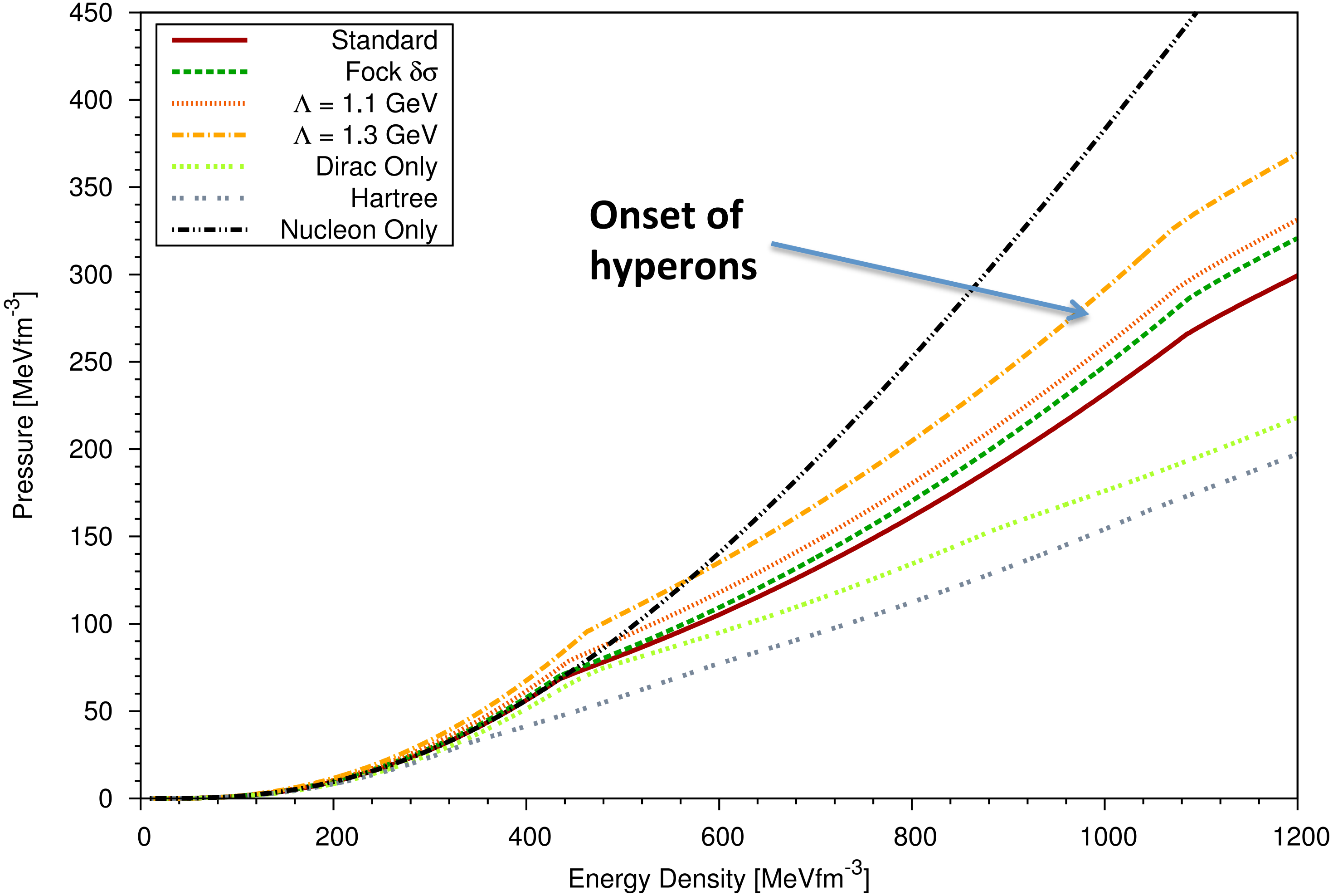


	$^{89}_{\Lambda}$Yb (Expt.)	$^{91}_{\Lambda}$Zr	$^{91}_{\Xi^0}$Zr	$^{208}_{\Lambda}$Pb (Expt.)	$^{209}_{\Lambda}$Pb	$^{209}_{\Xi^0}$Pb
$1s_{1/2}$	-22.5	-24.0	-9.9	-27.0	-26.9	-15.0
$1p_{3/2}$		-19.4	-7.0		-24.0	-12.6
$1p_{1/2}$	-16.0 (1p)	-19.4	-7.2	-22.0 (1p)	-24.0	-12.7
$1d_{5/2}$		-13.4	-3.1	—	-20.1	-9.6
$2s_{1/2}$		-9.1	—	—	-17.1	-8.2

Predicts Ξ bound by 10 – 15 MeV (to be tested in JPARC)

Increasing split between Λ and Ξ masses with increasing density.

Pressure as a function of energy density as predicted by QMC with hyperons

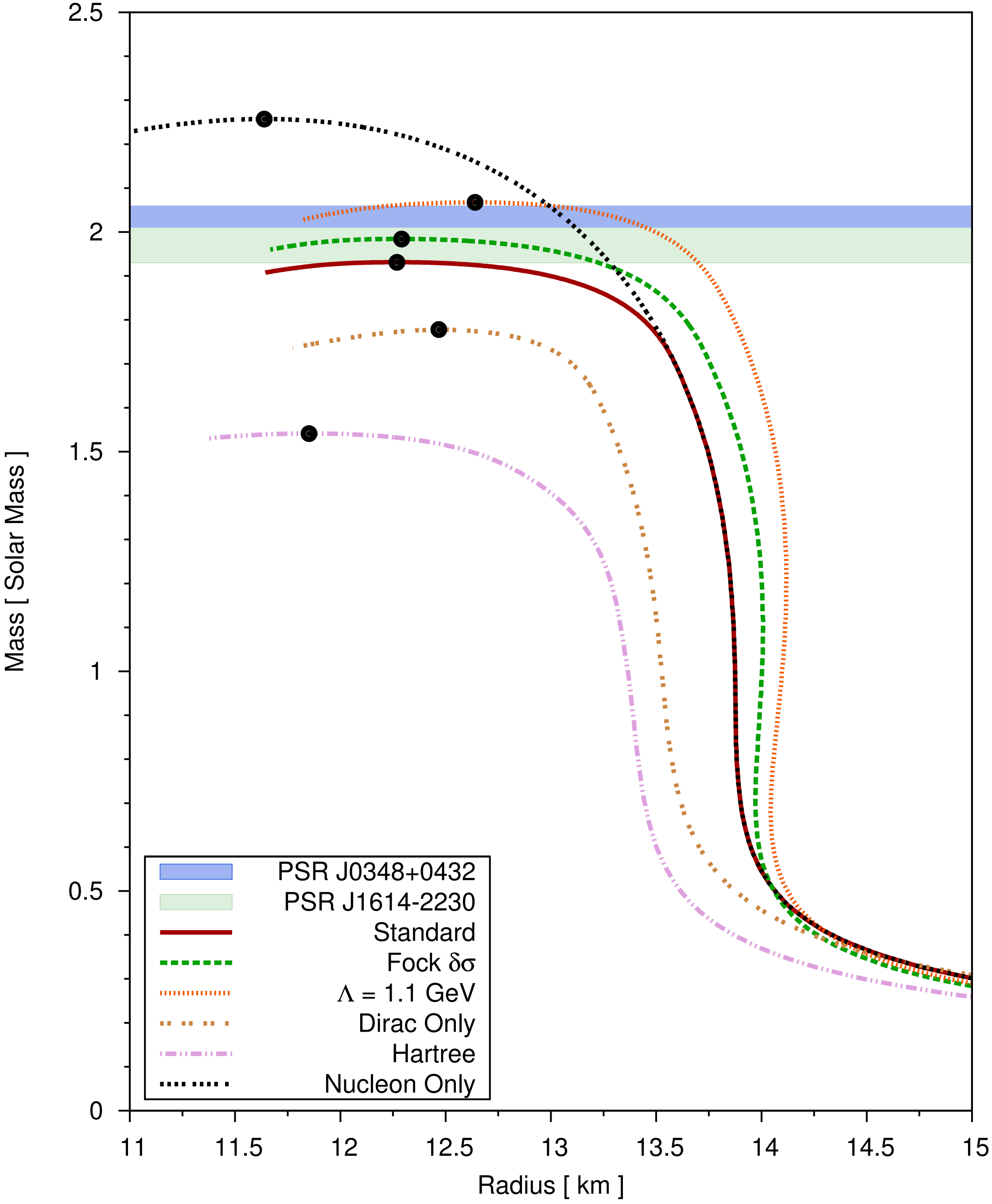


Onset of hyperons

Results: Cold neutron star

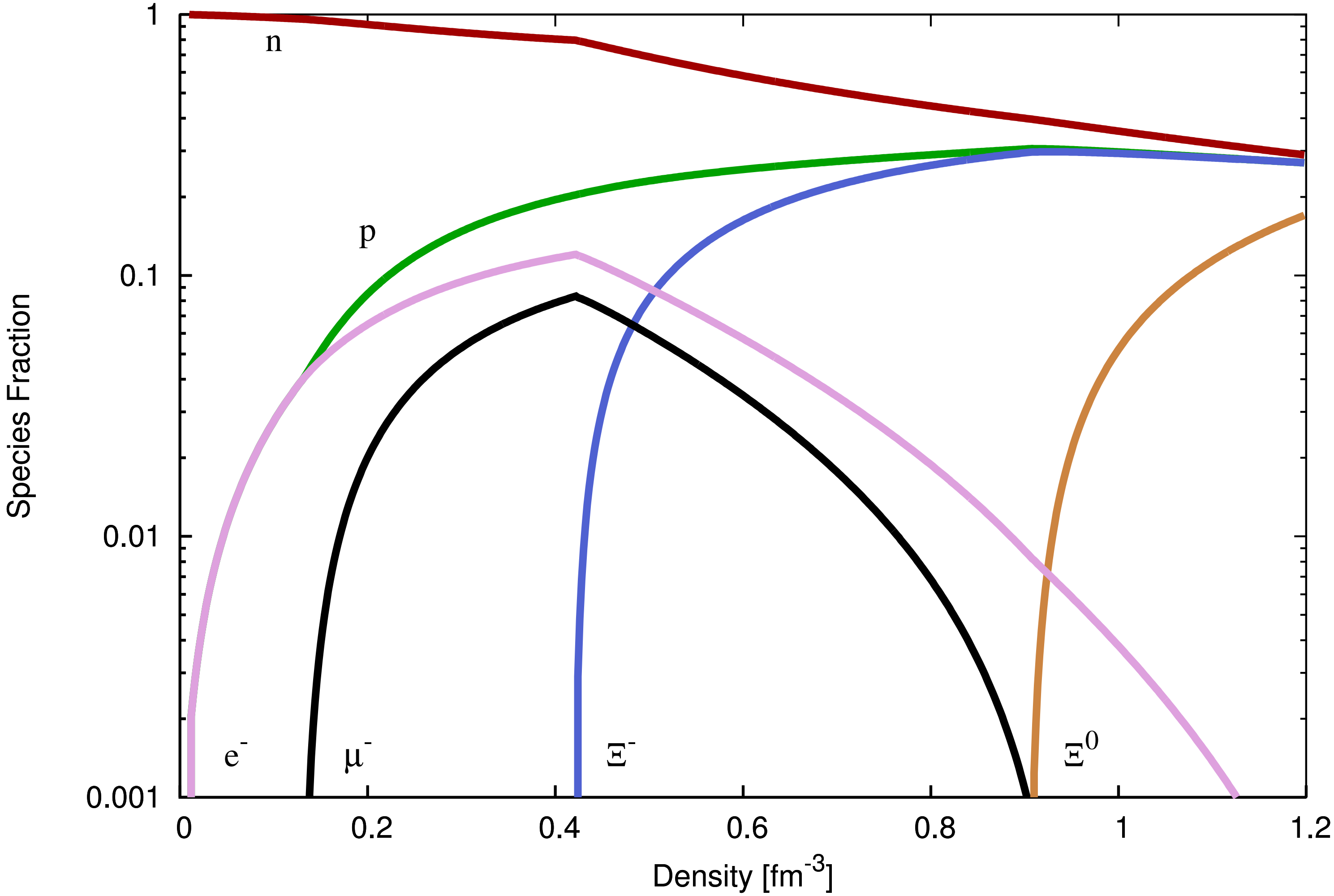
Model	$g_{\sigma N}$	$g_{\omega N}$	g_{ρ}	K_0 (MeV)	L (MeV)	R (km)	$M_{\max}$ ($M_{\odot}$)	$\rho_c^{\max}$ (ρ_0)
Standard	10.42	11.02	4.55	298	101	12.27	1.93	5.52
$\Lambda = 1.0$	10.74	11.66	4.68	305	106	12.45	2.00	5.32
$\Lambda = 1.1$	11.10	12.33	4.84	312	111	12.64	2.07	5.12
$\Lambda = 1.2$	11.49	13.06	5.03	319	117	12.83	2.14	4.92
$\Lambda = 1.3$	11.93	13.85	5.24	329	124	13.02	2.23	4.74
$R = 0.8$	11.20	12.01	4.52	300	110	12.41	1.98	5.38

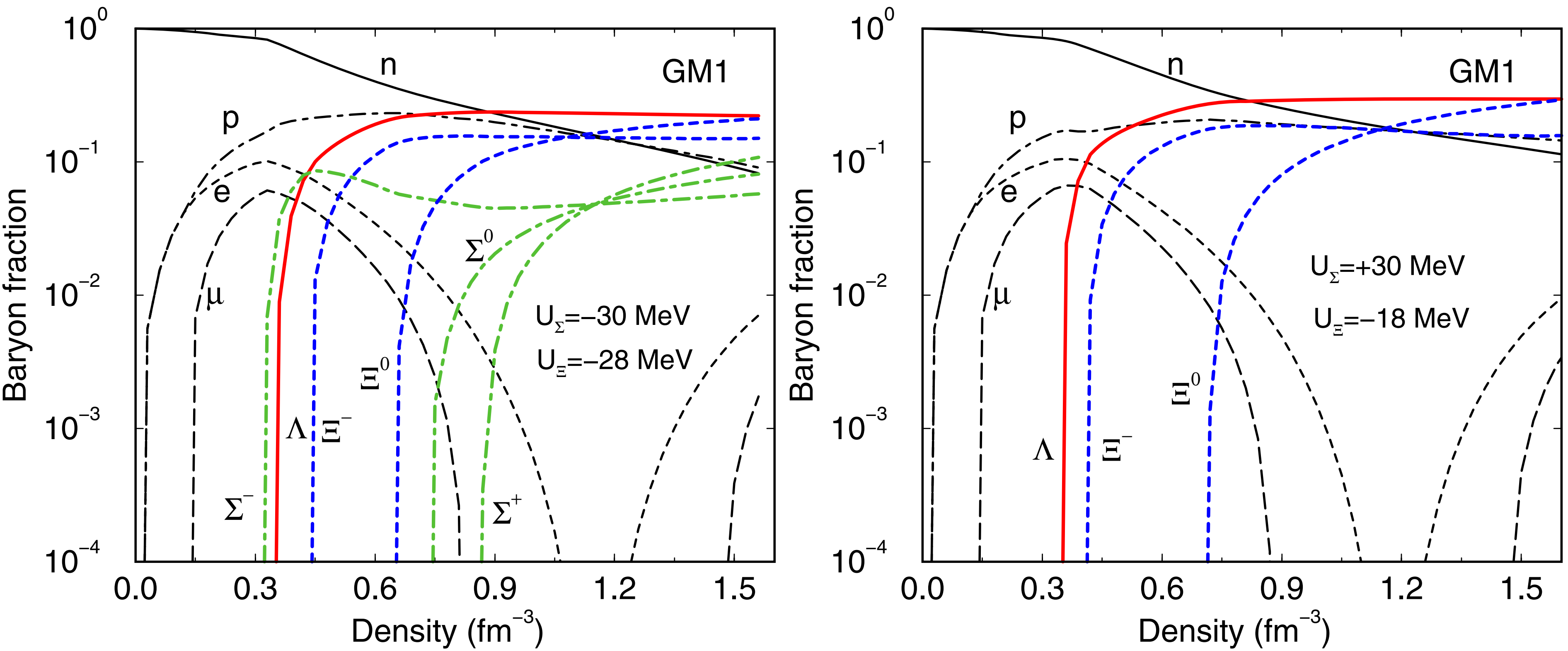
Stone, Stone and Moszkowski: accepted in PRC: $250 < K_0 < 315$ MeV



Antoniadis et al.
Demorest et al.

QMC predicted composition of HD matter (Y-N potentials calculated)





RMF with GM1 interaction empirical Y-N potentials fitted selfconsistently to data

J. Schaeffner-Bielich, NPA 835, 279 (2010)

Application to finite nuclei:

Guichon, Matevosyan, Sandulescu, Thomas, NPA 772, 1, 2006

Density dependent force in a non-relativistic approximation can be derived form QMC. The Hamiltonian depends on QMC coupling constants and polarizability **d** but has formally similar structure to the Skyrme forces.

$$\mathcal{H}_0 + \mathcal{H}_3 = \rho^2 \left[\frac{-3\,G_\rho}{32} + \frac{G_\sigma}{8\,(1+\enclose{circle}{d}\rho\,G_\sigma)^3} - \frac{G_\sigma}{2\,(1+\enclose{circle}{d}\rho\,G_\sigma)} + \frac{3\,G_\omega}{8} \right] + (\rho_n - \rho_p)^2 \left[\frac{5\,G_\rho}{32} + \frac{G_\sigma}{8\,(1+\enclose{circle}{d}\rho\,G_\sigma)^3} - \frac{G_\omega}{8} \right],$$

○ **highlights**
scalar polarizability

Guichon, Matevosyan, Sandulescu, Thomas, NPA 772, (2006)

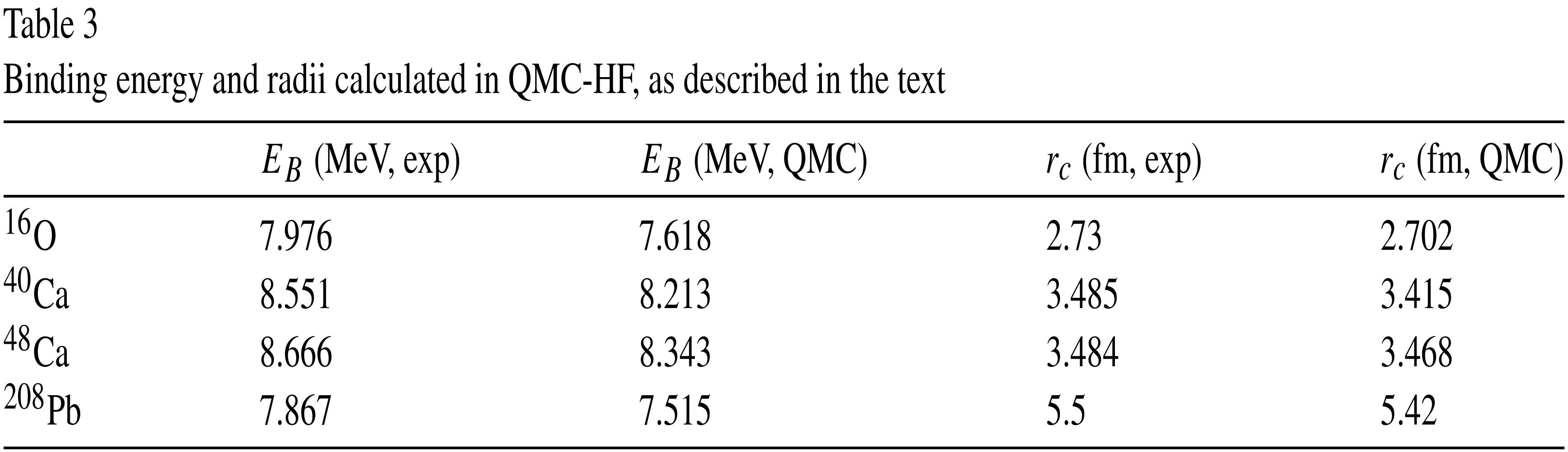


Table 3

Binding energy and radii calculated in QMC-HF, as described in the text

	E_B (MeV, exp)	E_B (MeV, QMC)	r_c (fm, exp)	r_c (fm, QMC)
^{16}O	7.976	7.618	2.73	2.702
^{40}Ca	8.551	8.213	3.485	3.415
^{48}Ca	8.666	8.343	3.484	3.468
^{208}Pb	7.867	7.515	5.5	5.42

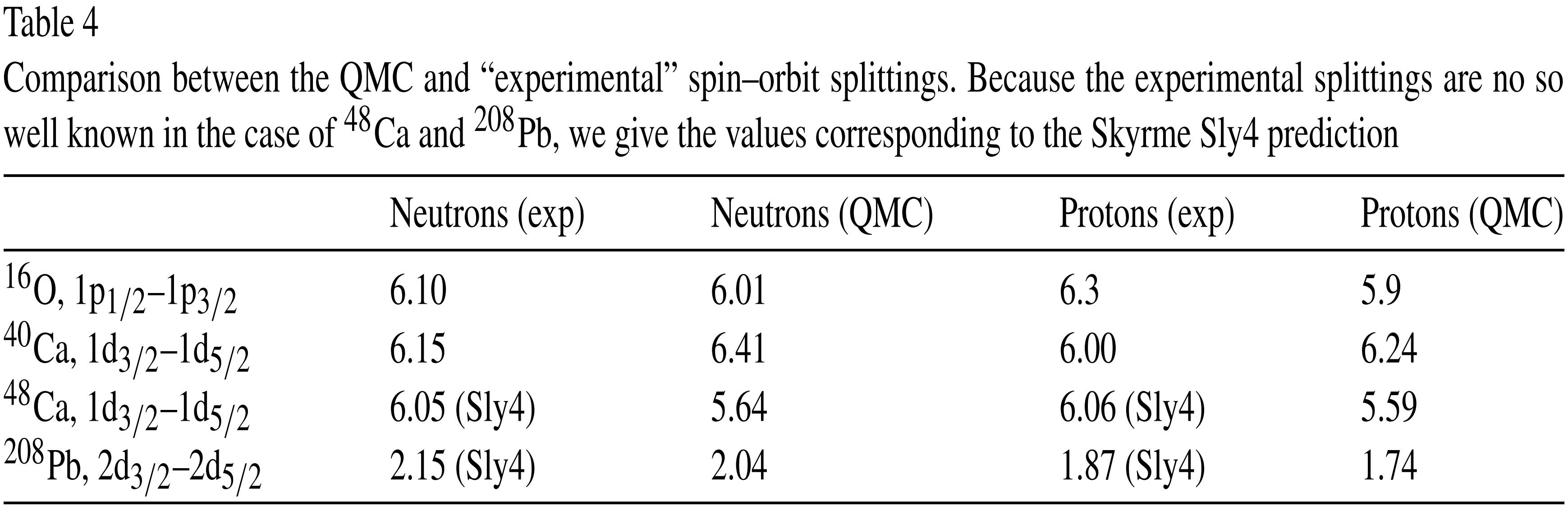
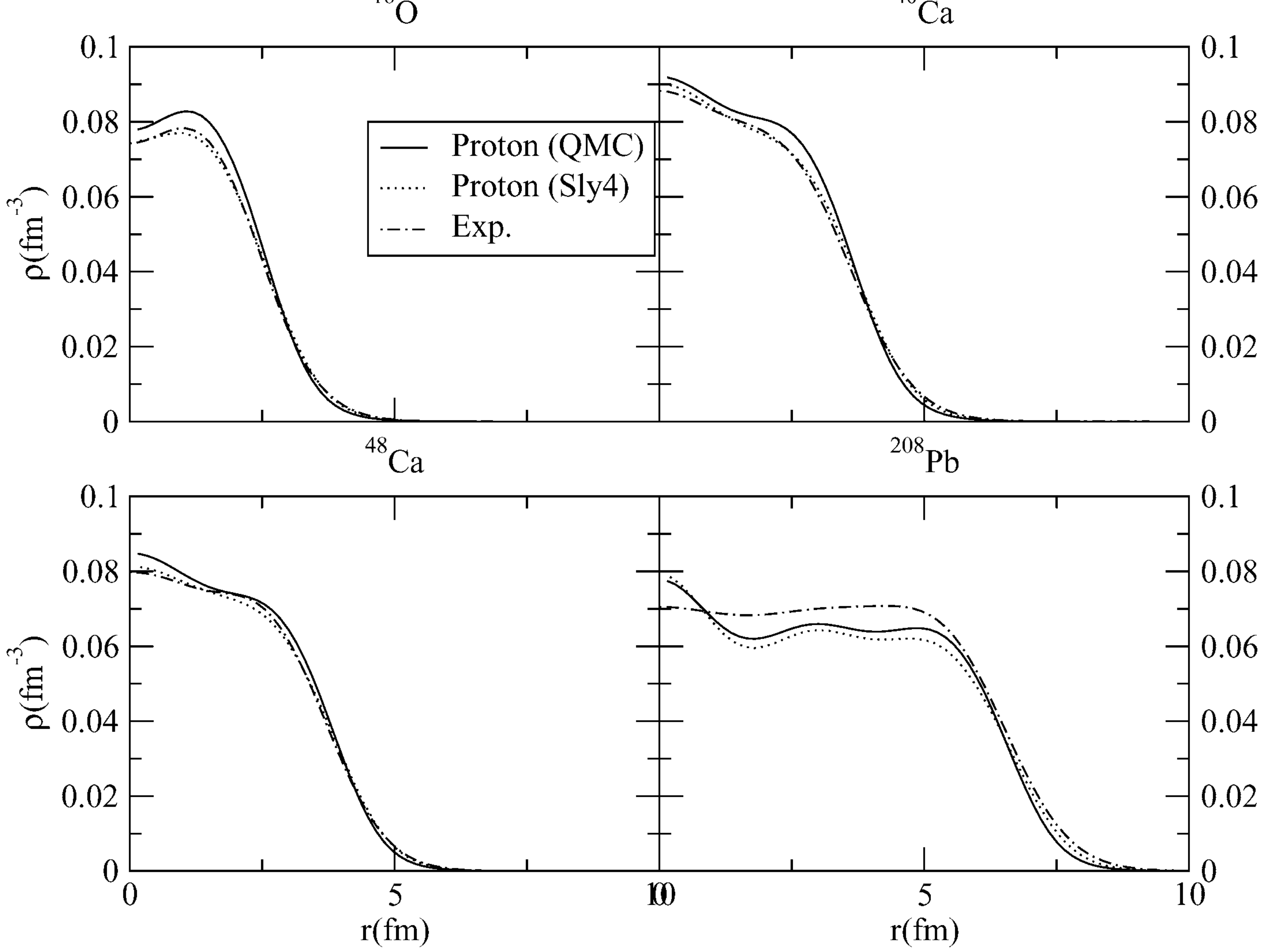


Table 4

Comparison between the QMC and “experimental” spin–orbit splittings. Because the experimental splittings are no so well known in the case of ^{48}Ca and ^{208}Pb, we give the values corresponding to the Skyrme Sly4 prediction

	Neutrons (exp)	Neutrons (QMC)	Protons (exp)	Protons (QMC)
^{16}O, $1p_{1/2}$–$1p_{3/2}$	6.10	6.01	6.3	5.9
^{40}Ca, $1d_{3/2}$–$1d_{5/2}$	6.15	6.41	6.00	6.24
^{48}Ca, $1d_{3/2}$–$1d_{5/2}$	6.05 (Sly4)	5.64	6.06 (Sly4)	5.59
^{208}Pb, $2d_{3/2}$–$2d_{5/2}$	2.15 (Sly4)	2.04	1.87 (Sly4)	1.74



QMC proton density distribution compared with experiment and Skyrme SLy4

QMC has a natural explanation for saturation of nuclear matter and in-medium effects through many-body forces

It is not limited to nucleons but can be applied to hyperons and CALCULATE interaction of any hadron in nuclear medium with NO ADDITIONAL parameters.

Yields effective, density dependent Λ N, Σ N, Ξ N forces (not yet published) with NO additional parameters – reproduces known properties of hypernuclei

Can be used to derive density-dependent effective force such as the Skyrme force which performs well in finite nuclei (HF+BCS QMC code for axially symmetric nuclei has been just developed and is in a testing stage (with P. - G. Reinhard)

BUT

IF QMC is as valid as we believe, it has to yield predictions consistent with results in other areas of nuclear physics and astrophysic

FUTURE: EoS for supernova matter (Chikako Ishizuka, Akira Ohnishi)
(QMC at finite temperature)

Statistical analysis of mass and radii of NS (Andrew Steiner)

Projected shell model (Yang Sun in Shanghai)

Ab-initio calculation of light nuclei (Emiko Hiyama at RIKEN)

Rotating neutron stars (Fridolin Weber + collaborators)

\+ + +

SUGGESTIONS WELCOME

SUMMARY

I. We do not understand behaviour of hadrons in dense medium.

II. Current models have limited predictive power – they have too many parameters and it is impossible to constrain them unambiguously

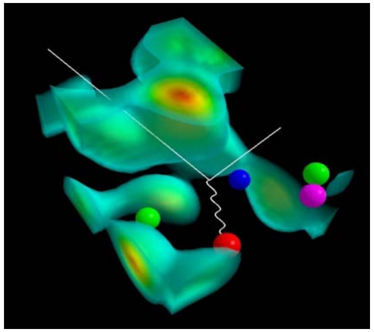
III. Models are often adjusted to fit only a selected class of data well, but they failure elsewhere is neglected . Such models cannot be right. Even "minimal" models are of a limited use in a broader context.

POSSIBLE SOLUTION?

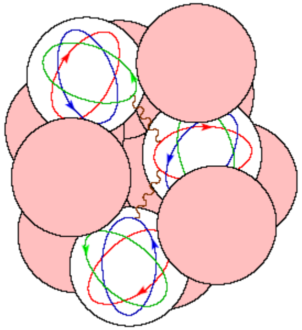
Evaluate basic assumptions of each models and regions of applicability
Focus on models with INDIVIDUAL parameters constrained by physics
Microphysics is important!

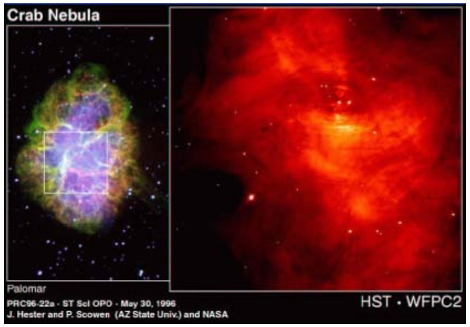
DATA LIMITED BY AVAILABLE TECHNIQUE – PHYSICS SHOULD BE ADOPTED AS A CONSTRAINT

N, Λ, Ξ, ω, **D,**
J/Ψ in
nuclear matter



QCD & hadron structure

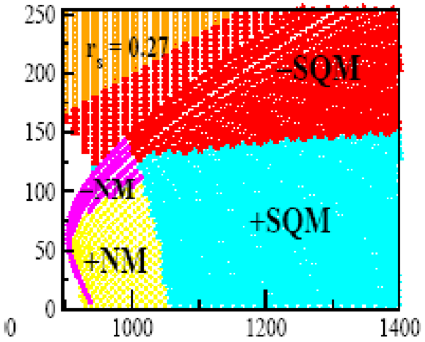




n star

∞ nuclear matter

Density dependent effective NN (and N Λ, N Ξ) forces



quark matter

Structure of finite nuclei & hypernuclei

THE UNIVERSITY of ADELAIDE

Courtesy Anthony Thomas University of Adelaide

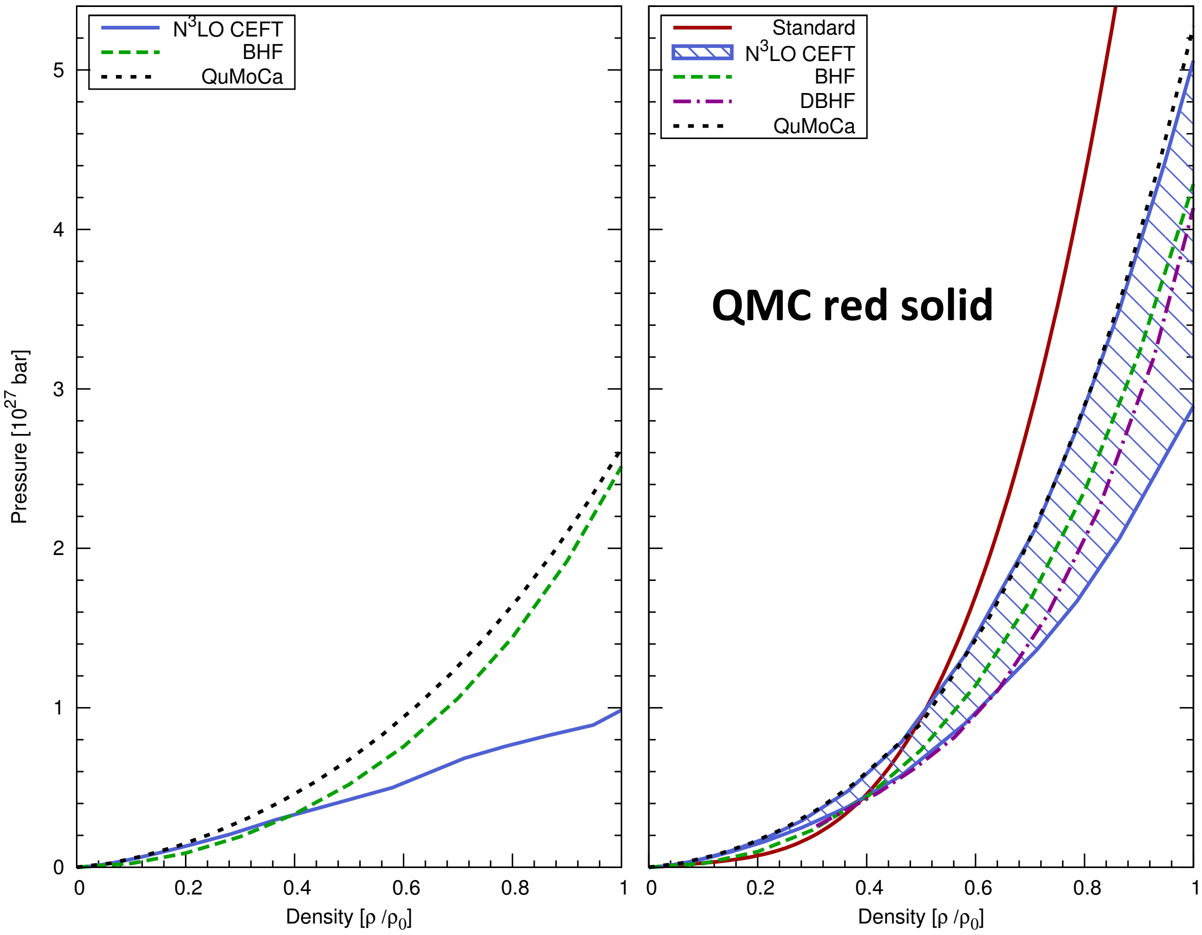
SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

Back-up slides

Pressure in pure neutron matter as calculated in different models

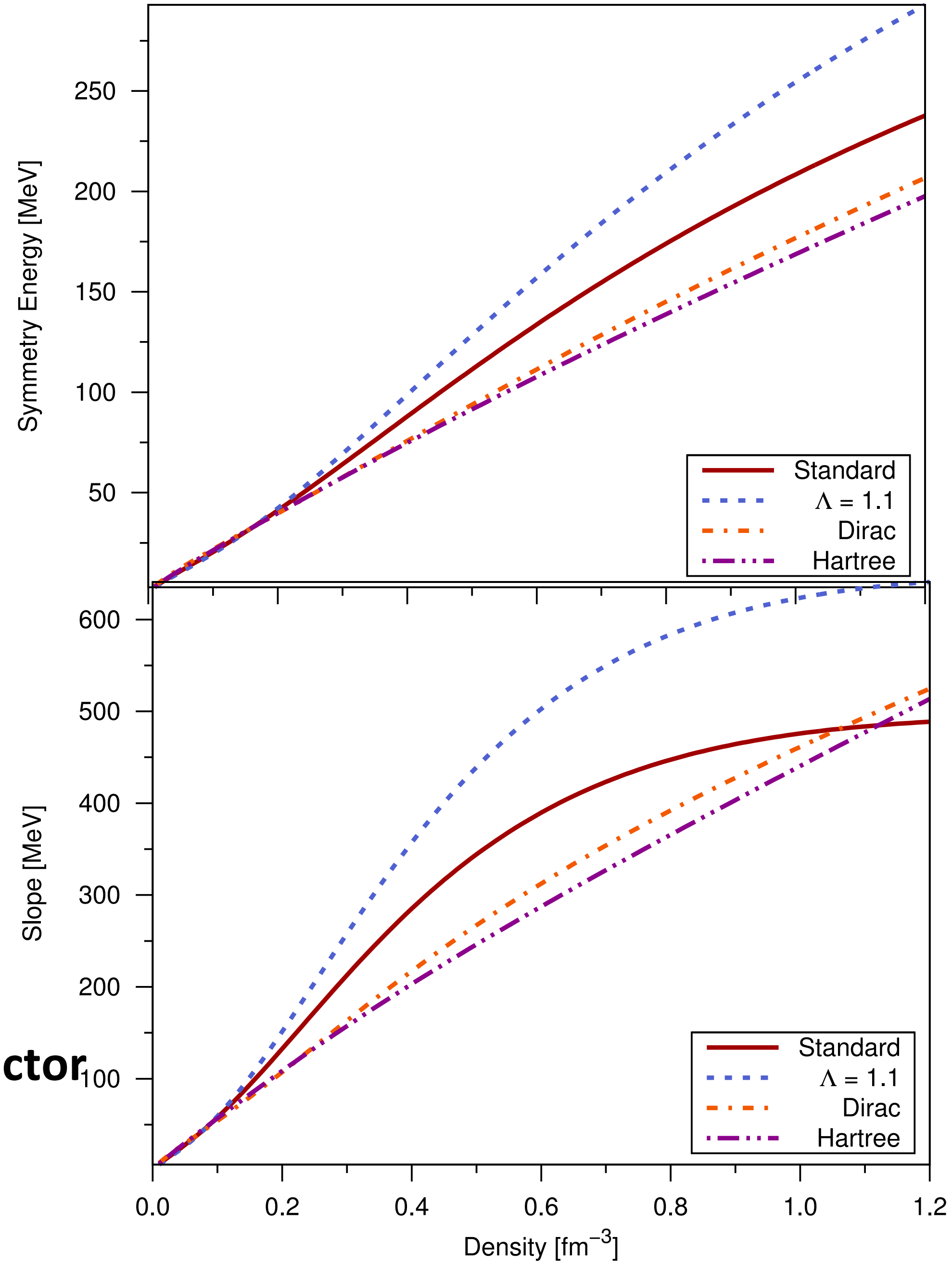
Left panel: without 3BF **Right panel: the same but with 3BF.**

DBHF added in right panel [Tsang et al., PRC 86, 015803 (2012)]



QMC red solid

Symmetry energy S (top) and its slope L (bottom) as a function of baryon number density as calculated in QMC.



Effect of the Fock term:

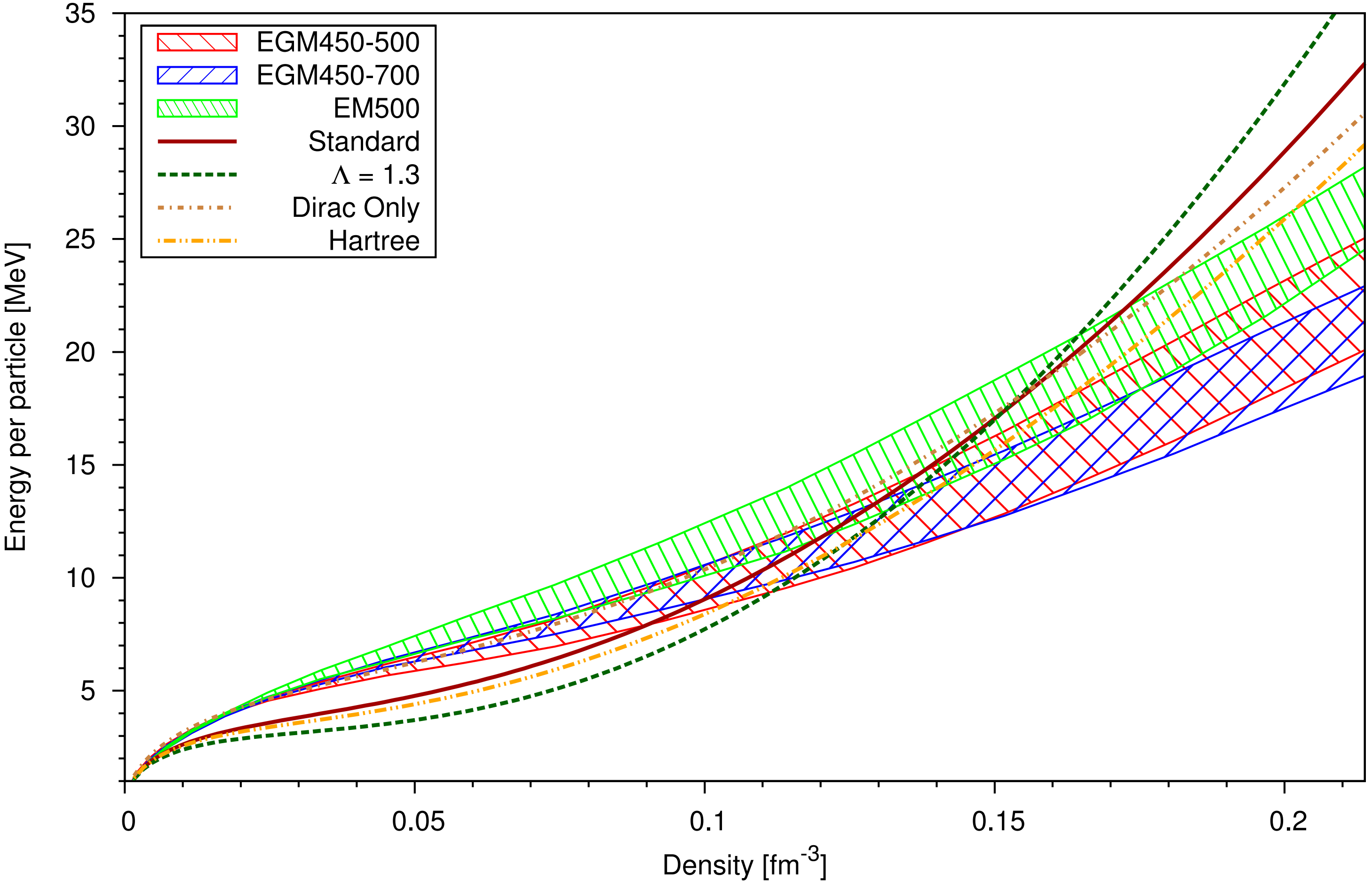
Standard: vector + tensor

Dirac: vector

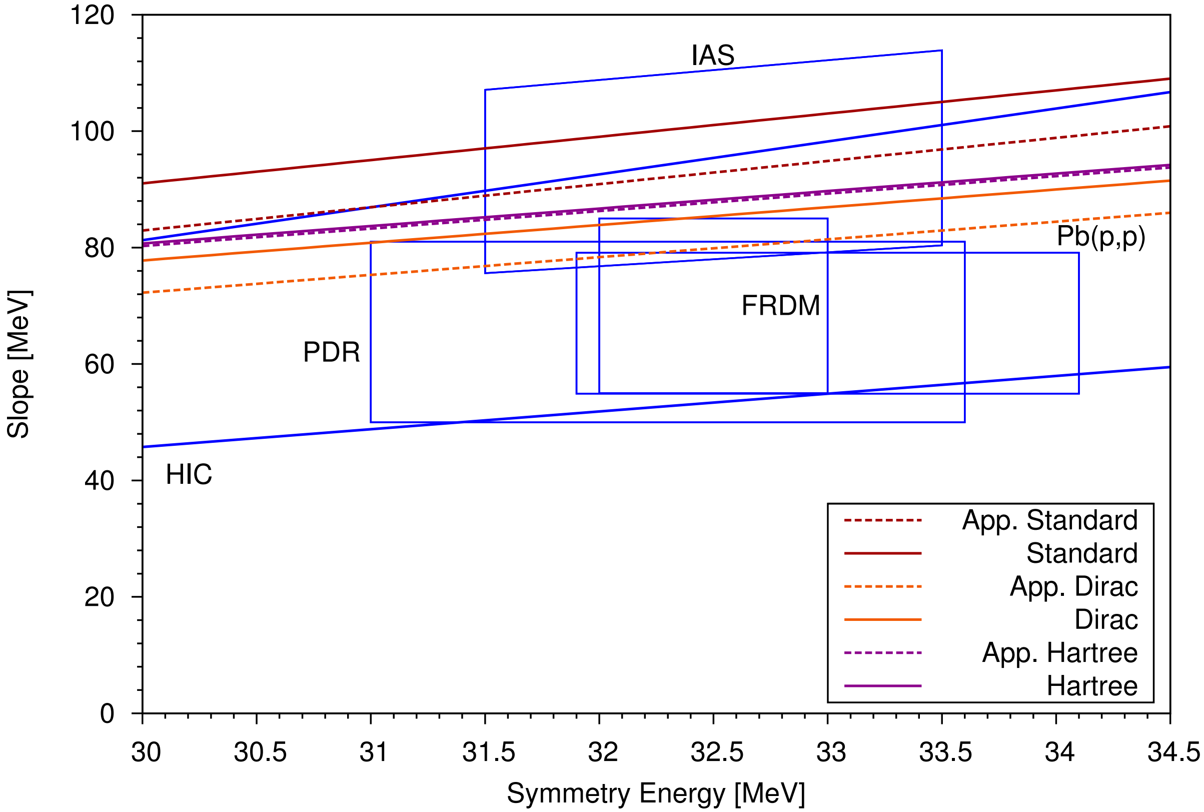
Hartree: no Fock term

Λ cut-off parameter of the form-factor in the Fock term.

Pure neutron matter energy per particle as a function of density as obtained in QMC, in comparison with complete CEFT at N^{3}LO order for more details of the latter see: *I. Tews, T. Krueger, K. Hebeler and A. Schwenk, Phys. Rev. Lett. 110 (2013) 032504*



Updated constraints Tsang et al., PRC 86, 015803 (2012)



$$\mathcal{S}(\rho) = \frac{1}{2}\frac{\partial^2 E}{\partial \beta^2}\Big|_{\rho,\beta=0},$$

$$E = \epsilon_{\text{hadronic}}/\rho,$$

“App”

$$S(\rho) = \mathcal{E}(\rho,\beta=1) - \mathcal{E}(\rho,\beta=0) ,$$

$$\mathcal{E} = \frac{1}{\rho}\left(\epsilon_{\text{hadronic}} - \sum_{B} M_B\rho_B\right)$$